

Mathematical Appendix

This appendix provides a short review of a number of concepts from algebra and geometry, as well as an introduction to differential calculus and Excel from a user's perspective. Almost all college students have had algebra and geometry classes in high school, but often this material is "lost" because it has not been used for years. Unlike algebra and geometry, many students have not used Excel or taken a class in calculus. Nonetheless, both are useful tools for learning microeconomics. Excel is basically a spreadsheet program that allows the user to do repetitive calculations easily. Excel is not a traditional part of learning microeconomics, but it makes economic modeling easier because it can be used for comparative statics analysis. Anyone who has passed an introductory microeconomics class already knows some applied calculus because the concept of "marginal analysis" that is preeminent in microeconomics is "economics-speak" for the calculus concept of derivative. As a result, the basic rules of differential calculus are laid out from a user's perspective.

MA.1 A Review of Algebra and Geometry

This section does not provide an exhaustive review of algebra and geometry. Rather, it is a compilation of algebraic and geometric concepts that have often been forgotten and that are useful for microeconomic analysis.

Order of Operations and Basic Algebra Rules

When equations are written, the order in which the various parts of the function should be implemented is well defined. For example, if $x = 2 + 3 \cdot 4$, then $x = 14$, not 20, because multiplication (of 3 times 4) has priority over (occurs before) addition and subtraction, even though the addition sign is to the left of the multiplication sign. By contrast, $y = 2 \cdot (3 + 4)$ equals 14, not 10, because parentheses have priority over multiplication.

The hierarchy of operations is as follows:

1. Operations are performed from left to right, subject to the rules that follow.
2. Parentheses have the highest priority. Inner parentheses have priority over outer parentheses.
3. Exponentiation has the next highest priority.
4. Multiplication and division have the next highest priority.
5. Addition and subtraction have the lowest priority.

As an equation is evaluated from left to right, it is important to maintain the order of operations just listed. If these rules are not followed, then an incorrect answer is the inevitable result.

Chapter Outline

- MA.1** A Review of Algebra and Geometry
- MA.2** A Primer on Differential Calculus
- MA.3** Using Excel in Economic Modeling

Section MA.1 Outline

- Order of Operations and Basic Algebra Rules
- Univariate Functions
- Multivariate Functions
- Linear Models
- Quadratic Models and the Quadratic Formula
- Exponential and Logarithmic Models
- Making Economic Sense of Mathematical Models

When writing equations, students sometimes add parentheses because they are concerned that the order of operations will not work. For example, the “store method” of the budget constraint described in Equation 5.1a is based on how you purchase goods, given income I and prices P_x and P_y ; $P_x \cdot x$ is spent on x , and $P_y \cdot y$ is spent on y , so $I \geq P_x \cdot x + P_y \cdot y$. If all income is spent, $I = P_x x + P_y y$. (We should note in this equation that the multiplication of P_x and x is implicit in the equation. Whenever an equation has two variables next to one another, there is an implicit multiplication sign between those variables. The more formal version of this equation is: $I = P_x \cdot x + P_y \cdot y$.) Students often write this as $I = (P_x x) + (P_y y)$ because they are concerned that the addition will occur before the second multiplication (because operations are done from left to right). But multiplication has priority over addition, so the parentheses are not necessary. On the other hand, the parentheses are not harmful—they are simply redundant. By understanding order of operations, you obtain the correct answer when calculating an equation with multiple operations, and you also can produce more parsimonious equations that represent the calculations you wish to undertake.

The following facts from algebra are worth remembering because they often come up in problem solving. Many of these rules are so common that you probably have forgotten their names.

Facts from Algebra		
Addition and Multiplication	<i>Of Addition</i>	<i>Of Multiplication</i>
<i>Commutative Property</i>	$a + b = b + a.$	$a \cdot b = b \cdot a.$
<i>Associative Property</i>	$a + (b + c) = (a + b) + c.$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
<i>Identities with Zero</i>	$0 + a = a.$	$0 \cdot a = 0.$
		If $a \cdot b = 0$, then $a = 0$, or $b = 0$, or both.
<i>Identities with One</i>		$1 \cdot a = a.$
<i>Minus Signs, Single</i>	$a + (-b) = a - b.$	$(-a) \cdot b = a \cdot (-b) = -a \cdot b.$
<i>Double</i>	$a - (-b) = a + b.$	$(-a) \cdot (-b) = a \cdot b.$
<i>Inverses</i>	$a + (-a) = 0.$	$a \cdot a^{-1} = a/a = 1$ for $a \neq 0.$
<i>Distributive Property of Multiplication over Addition</i>		$a \cdot (b + c) = a \cdot b + a \cdot c.$
<i>Subtraction Signs Distribute through Parentheses</i>		$a - (b + c) = a - b - c.$
Division (rules for fractions)		
<i>Common Denominator, Addition, $c \neq 0$</i>		$a/c + b/c = (a + b)/c.$
<i>Common Denominator, Subtraction, $c \neq 0$</i>		$a/c - b/c = (a - b)/c.$
<i>Different Denominators, Addition, $c \neq 0$, and $d \neq 0$</i>		$a/c + b/d = (a \cdot d + b \cdot c)/(c \cdot d).$
<i>Different Denominators, Subtraction, $c \neq 0$, and $d \neq 0$</i>		$a/c - b/d = (a \cdot d - b \cdot c)/(c \cdot d).$
<i>Multiplication, $c \neq 0$, and $d \neq 0$</i>		$(a/c) \cdot (b/d) = (a \cdot b)/(c \cdot d).$
<i>Division, $b \neq 0$, $c \neq 0$, and $d \neq 0$</i>		$(a/c)/(b/d) = (a \cdot d)/(c \cdot b).$
<i>Rationalizing a Fraction, $a \neq 0$, and $b \neq 0$</i>		$1/(a/b) = b/a.$
<i>Multiplication by 1 ($1 = c/c$), $b \neq 0$, and $c \neq 0$</i>		$(a/b) \cdot (c/c) = a/b$

Power rules are discussed later in this section (see Equations MA.8f–MA.8n).

Common Mistakes

	<i>Mistaken “Rule”</i>	<i>Examples</i>
Subtraction is <i>not</i> commutative.	$a - b \neq b - a.$	$2 - 1 = 1 \neq -1 = 1 - 2.$
Division is <i>not</i> commutative.	$a/b \neq b/a.$	$2/1 = 2 \neq 0.5 = 1/2.$
Subtraction is <i>not</i> associative.	$a - (b - c) \neq (a - b) - c.$	$1 - (2 - 3) = 2 \neq -4$ $= (1 - 2) - 3.$
Division is <i>not</i> associative.	$a/(b/c) \neq (a/b)/c.$	$6/(2/3) = 9 \neq 1 = (6/2)/3.$

Exponents do not distribute over addition: $(a + b)^c \neq a^c + b^c$ unless $c = 1$. In particular, $(a + b)^2 \neq a^2 + b^2$. In fact, $(a + b)^2 = a^2 + 2 \cdot a \cdot b + b^2$.

Similarly, the square-root function does not work this way: $(a + b)^{0.5} \neq a^{0.5} + b^{0.5}$. To see this, note that $a = 9$ and $b = 16$ produces $(9 + 16)^{0.5} = 5 \neq 7 = 9^{0.5} + 16^{0.5}$.

Strategies to Solve Problems

There are many ways to solve algebraic problems, and it would be impractical to present a definitive list of methods here. Nonetheless, the following problem-solving tactics are common:

Addition: If we add (or subtract) the same number to both sides of an equation, we obtain an equivalent equation. Algebraically, we have: If $a = b$, then $a + c = b + c$.

Example: $L + 16 = 24$. Subtract 16 from both sides to obtain: $L = 24 - 16 = 8$.

Multiplication: If we multiply (or *divide*) the same nonzero number to both sides of an equation, we obtain an equivalent equation. Algebraically, we have: If $a = b$, then $a \cdot c = b \cdot c$.

Example: $3 \cdot x = 24$. Divide both sides by 3 to obtain: $x = 24/3 = 8$.

Equations involving fractions:

1. Denominators cannot equal zero, and therefore, values of a variable that produce zero in the denominator must be excluded.
2. Multiply by the common denominator to obtain an equivalent equation without fractions.

Example: Find x that satisfies $2/(x - 4) + 1/x = -x/(x - 4)$. x cannot equal 0 or 4, since this results in a zero in the denominator. The least-common denominator is $x \cdot (x - 4)$, so multiply both sides by $x \cdot (x - 4)$. This removes each of the denominators and yields the following equivalent equation: $2 \cdot x + x - 4 = -x^2$. Add x^2 to both sides to produce a quadratic equal to zero: $x^2 + 3 \cdot x - 4 = 0$. Factor to obtain: $(x + 4) \cdot (x - 1) = 0$. Therefore, $x = -4$, or $x = 1$. If you cannot see a factorization, use the quadratic formula (Equation MA.7) to obtain the same result. Finally, you should check to make sure that these values of x satisfy the equation: $x = -4$ (left-hand side: $2/(-4 - 4) + 1/(-4) = -1/2$; right-hand side: $-(-4)/(-4 - 4) = -1/2$), and $x = 1$ (left-hand side: $2/(1 - 4) + 1/1 = 1/3$; right-hand side: $-1/(1 - 4) = 1/3$).

Equations involving parentheses: Expand parentheses and regroup.

Example: $2 \cdot (x - 2) = 20$. Expand to obtain: $2 \cdot x - 4 = 20$, $2 \cdot x = 24$, $x = 12$.

Multiple equations in multiple unknowns: Various methods are used to solve multiple equations, but the simplest is to solve for one of the variables in terms of the others and then substitute this equation into the other equations. This reduces the number of unknown variables.

Example: Find x and y that satisfy: $x - y = 2$, and $2 \cdot x + y = 13$. Solve for y in the first equation: $y = x - 2$. Substitute into the second equation to obtain: $2 \cdot x + (x - 2) = 13$. Regroup to obtain: $3 \cdot x = 15$. Divide by 3 to obtain: $x = 5$. Substitute into $y = x - 2$

to obtain: $y = 3$. You should check to make sure that $(x, y) = (5, 3)$ satisfies both equations: $5 - 3 = 2$, and $2 \cdot 5 + 3 = 13$.

Univariate Functions

A *function* is a way of describing a relationship between two variables. We typically write this as:

$$y = f(x). \quad (\text{MA.1})$$

Each value of x is associated with a single value of y : y is known as the *dependent variable*, and x is the *independent variable*. This is called a *univariate function*. The most common such relationship in microeconomics is demand. In this context, y is the quantity of a good demanded, and x is the price of that good: Quantity demanded depends on price.

There are a number of ways to represent functional relationships. One method is to have a table listing a series of values of the independent variable, together with the associated values of the dependent variable. This tabular information can also be described as a series of *ordered pairs*, (x, y) . A second method is to describe the information graphically. A third method is to describe the information algebraically.

Suppose that we have the following information about x and y for four data points, **A–D**. Boldfaced letters denote ordered pairs in this context so that **C** = (2, 4) signifies that point **C** has 2 units of x and 4 units of y . The relation between x and y is straightforward: y is x squared. For many purposes, the smoothed graph and algebraic forms provide the simplest and most intuitive description of the data.

Given that we do not know the identity of variables x and y , it is only mathematical convention that defines y as the dependent variable and x as the independent variable. Mathematicians put the independent variable on the horizontal axis and the dependent variable on the vertical axis. Unfortunately, economists do not always follow this mathematical convention. The most well-known abuse of this convention is, of course, the demand curve. As noted earlier, quantity demanded depends on price: Quantity demanded is the dependent variable, and price is the independent variable. The demand graph, however, shows quantity on the horizontal axis and price on the vertical axis. One other economist idiosyncrasy should be pointed out here: Economists often use the term *curve* even when the curve is linear; the most common example is, once again, demand. Economists say things like: “Assume a linear demand curve” without worrying about the sensibilities of mathematicians who would consider this statement an oxymoron.

When a function is *one-to-one* (meaning that a given y is associated with at most one x), the function can be inverted. x is then written as a function of y . (To use the example in Table MA.1: If $y = x^2$, then $x = y^{0.5}$, since taking the square root of both sides of an equation produces an equivalent equation, but note that $y = x^2$ is only one-to-one for $x \geq 0$, since the (x, y) pairs $(-2, 4)$ and $(2, 4)$ both satisfy $y = x^2$.)

Often, there is an economic interpretation of this inverted function. Again, demand provides an example. Inverse demand describes marginal valuation as a function of quan-

TABLE MA.1 Data Organized in Various Ways

Ordered Pair	Table	Discrete Graph	Smoothed Graph	Algebraic
Point = (x, y)				$y = x^2$,
A = (0, 0)				given
B = (1, 1)				$x = 0, 1, 2, 3$
C = (2, 4)				for discrete x ,
D = (3, 9)				or
				$0 \leq x \leq 3.2$
				for continuous x .

tity (see Section 12.1). If, for example, $x = P^{-2}$ is a demand function, then inverse demand is $P = x^{-0.5}$, since $x = P^{-2}$ implies $x^{-0.5} = (P^{-2})^{-0.5} = P^{(-2 \cdot -0.5)} = P^1 = P$, using Equation MA.8n. This is, in fact, an example of a very useful demand function called constant elasticity demand (see Section 12.3).

When an increase in the independent variable x is associated with an increase in the dependent variable y , we say that y is *directly* related to x . When an increase in the independent variable x is associated with a decrease in the dependent variable y , we say that y is *indirectly* related to x . The function $y = x^2$ describes a direct relation between y and x for $x \geq 0$, and the graph of $P = x^{-0.5}$ is an indirect relation for $x > 0$. The most common way to recognize direct and indirect relations is by using slope.

Slope is simply change in the variable on the vertical axis for a given change in the variable on the horizontal axis:

$$\text{Slope} = \Delta y / \Delta x = (y_b - y_a) / (x_b - x_a). \quad (\text{MA.2})$$

The Greek letter *delta*, Δ , signifies difference. We often describe slope as rise over run. As we discuss in Section MA.2, slope is simply the calculus notion of derivative. (Calculus is composed of two broad parts: differential and integral calculus. Differential calculus studies slopes; integral calculus studies areas. Differential calculus is much more useful for microeconomics, although we will see applications of integral calculus in the discussion of welfare economics.) Lines and curves are distinguished by their slopes: The slope of a line is constant, and the slope of a curve varies.

A convenient way to check whether a function relating y and x is direct or indirect is to ask whether the numerator and denominator of the slope fraction are the same sign. If they are, the slope is positive, and the function is direct. This is the case for any pair of points chosen from points **A–D** in Table MA.1. By contrast, the inverse demand function $P = x^{-0.5}$ depicts an indirect relation between price and quantity (as you would expect with demand functions). (Sometimes, economists describe an indirect relation using a variation on the term *inverse*. In this conceptualization, y is *inversely related to* x , even though it is not *the* inverse function ($y = 1/x$.) This inverse demand function contains the (x, P) points **A** = (1, 1), **B** = (4, 0.5), and **C** = (16, 0.25). If you calculate the slope fraction between any pair of these points, you will note that the numerator and denominator *always* have opposing signs, or put another way, the slope is negative.

An increasing one-to-one function is called a *monotonic transformation*. The graph of such a function always has a positive slope; the slope of a monotonic function does not change signs. (Mathematicians sometimes are more explicit and call this a *positive* monotonic transformation because it is easy to distinguish this from a decreasing one-to-one function. We do not use negative monotonic transformations in this text.) The notion of monotonic transformation is central to the ordinality of the utility function discussed in Chapter 4.

A function $f(x)$ is *continuous* if there are no “jumps” in the function, meaning that you can draw it without lifting your pencil off the paper. (The formal definition requires that as x approaches a given value x_0 , then $f(x)$ must approach $f(x_0)$.) The function $f(x) = \text{Integer}(x)$ is not continuous at integer values of x . ($\text{Integer}(x)$ is a function that equals the integer part of a number.) In this instance, $f(x)$ is a step function with 1-unit-wide steps. Consider $x_0 = 1$; then $f(x_0) = 1$. If x approaches 1 from below, $f(x) = 0$, but if x approaches 1 from above, $f(x) = 1$. There is a jump *discontinuity* (step) at $x = 1$, $x = 2$, and so on. We work almost exclusively with continuous functions in this text.

A continuous function is *smooth* if the slope of the function at a point is the same, regardless of the direction in which the slope is viewed. Another way to describe a smooth function is to say that there is a unique tangent to any point on the function. Quite simply, a continuous function is smooth if it is not “pointy.” Most functions are smooth; the most common function that is *not* smooth is the absolute value function, $f(x) = |x|$. At $x_0 = 0$, the function has a slope of -1 if zero is approached from below (i.e., if $x \rightarrow 0$ with negative values of x), but it has a slope of $+1$ if zero is approached from above (i.e., if $x \rightarrow 0$ with positive values of x).

Multivariate Functions

A function need not be *univariate*. The term *univariate* means that there is only one independent variable. Demand is a function of both price and income. Demand is a *multivariate* function. There is only one dependent variable, but this variable depends on a number of independent variables, rather than only one. If y depends on n independent variables, then we would write this as:

$$y = f(\mathbf{X}) = f(x_1, \dots, x_n), \text{ where } \mathbf{X} = (x_1, \dots, x_n). \quad (\text{MA.3a})$$

The boldfaced \mathbf{X} denotes an ordered n -tuple.

As noted previously, univariate functions can be visualized with a two-dimensional graph. Multivariate functions require a greater number of dimensions. The function in Equation MA.3a requires $n + 1$ dimensions. Specifically, it is an n -dimensional surface in an $n + 1$ -dimensional space (good luck visualizing this). Often, we restrict our visual field to two independent variables. The univariate equation $z = x^2$ is a parabola. If we increase the dimensionality to include two independent variables x and y , then the equation $z = x^2 + y^2$ can be visualized as a paraboloid in (x, y, z) space. (This paraboloid is simply the parabola $z = x^2$ [shown in Table MA.1 with the vertical axis as z], spun around the z axis.) The production function $y = (L \cdot K)^{0.5}$ is similarly a rectangular hyperboloid (indeed, this is the equal-weighted Cobb-Douglas [CD] production function with constant returns to scale depicted in Figure 9.9B).

Three-dimensional graphs are difficult to draw (and the images take more space to load and store), so we often resort to two-dimensional projections of these three-dimensional surfaces. This is accomplished by setting the dependent variable equal to a series of constant values and mapping out the *level sets* (curves) that result. (This is exactly how the two- and three-dimensional graphs were derived in Figures 9.7, 9.9, and 9.10.) The result is an isoquant map in (L, K) space if a production function is being depicted or an indifference map in (x, y) space if a utility function is under consideration. Such maps are strictly analogous to the topographic maps that you may have used if you have ever been hiking. A topographic map has contour lines that join points of equal elevation; such contours depict hills and valleys.

Note: *We will often consider the hiking analogy as we examine the properties of functions.*

A multivariate function can be conceptualized in univariate terms by holding all independent variables save one constant. Suppose that we hold x_2, \dots, x_n constant in Equation MA.3a. In this event, y is a function of x_1 . Such a function is written as:

$$y = f(x_1; x_2, \dots, x_n). \quad (\text{MA.3b})$$

All variables to the right of the semicolon in Equation MA.3b are parametrically fixed (in this instance, x_2, \dots, x_n are fixed), and all variables to the left of the semicolon are variable (in this instance, x_1 is allowed to vary). This is how the total product curve in Figure 9.1 is embedded in the production function in Figure 9.12 ($TP(L) = Q(L; K)$, given $K = 10$). We often denote the fixed parameters using a superscript or subscript (such as $Q(L; K_0)$) or by noting explicitly the value of the parameter (such as $TP(L; K = 10)$). To use the hiking analogy, a total product curve is like hiking in a single direction; for example: $TP(K; L)$ is like hiking north and south in (L, K) space, holding the east/west coordinates (longitude or *labor*) fixed, while $TP(L; K)$ is like hiking east and west in (L, K) space, holding the north/south coordinates (latitude or *capital*) fixed.

Dependent variables are variables that you wish to “explain,” using other independent variables. Sometimes, you cannot control an independent variable; such variables are *exogenous* to the system. If you can control a variable, it is *endogenous* to the system. If you own an ice-cream parlor, you are likely to realize that the demand for ice cream depends on both price and temperature. Price is endogenous (as long as you have some price-setting ability—that is, ice-cream parlor ice cream is not perfectly competitive in

your town), but temperature is exogenous. Demand depends negatively on price and positively on temperature. If we denote the price of ice cream by P , the quantity of ice cream by C , and the daily high temperature by T , then we can write this as:

$$C = f(P, T). \quad (\text{MA.4a})$$

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The plus and minus signs beneath each independent variable denote the expected direction of change in C as each variable increases. This same expectation can be written using slopes. Slope is the change in the dependent variable for a given change in an independent variable. The expectations in Equation MA.4a may therefore be rewritten as:

$$\Delta C/\Delta P < 0, \text{ and } \Delta C/\Delta T > 0. \quad (\text{MA.4b})$$

These slopes assume that the values of all other independent variables remain fixed. This is an example of the economic notion of *ceteris paribus*, and mathematically, it is also the notion of a *partial derivative*, which we discuss in Section MA.2. To use the hiking analogy, $\Delta C/\Delta P$ is the steepness of the C function in the P direction, holding T fixed, and $\Delta C/\Delta T$ is the steepness of the C function in the T direction, holding P fixed.

Algebra and Geometry Example 1: Slopes in Different Directions

Facts: Suppose that you manage an ice-cream shop, and you find that demand for ice-cream cones C on a given day is approximately given by: $C(P, T) = 250 - 100 \cdot P + 5 \cdot T$, where P is the price per cone, and T is the high temperature for the day.

- How many cones do you expect to sell if you charge \$2 per cone and the temperature reaches 90° ?
- What are $\Delta C/\Delta P$ and $\Delta C/\Delta T$ at this (P, T) point, and what is the economic interpretation of each slope?

Solution:

- Substitute $P = 2$ and $T = 90$ into $C(P, T)$ to obtain an estimate of unit sales in this instance:

$$C(2, 90) = 250 - 100 \cdot 2 + 5 \cdot 90 = 250 - 200 + 450 = 500.$$

You would expect to sell 500 ice-cream cones on this day.

- The coefficient attached to each independent variable is the slope in that direction, given a linear model.

If you did not recognize that this is a linear model, you can easily calculate slope, using Equation MA.2. To find slope in the P direction, increase P by 1 and see what happens to sales:

$$C(3, 90) = 250 - 100 \cdot 3 + 5 \cdot 90 = 250 - 300 + 450 = 400.$$

Therefore, we see that slope in the P direction is:

$$\Delta C/\Delta P = (C(3, 90) - C(2, 90))/(3 - 2) = (400 - 500)/1 = -100/1 = -100.$$

Notice that this calculation holds temperature fixed. A symmetric calculation based on the (P, T) point $(2, 91)$ yields a slope in the temperature direction of $\Delta C/\Delta T = 5$.

Since this is a linear model, it does not matter what size ΔP or ΔT is used in calculating slope. Any size “run” will produce the same slope, given the linearity of the function $C(P, T)$.

These slopes suggest the change in sales for a given change in price or temperature:

If price increases by \$1, sales will decline by 100 cones, *ceteris paribus*.

If temperature increases by 1 degree, sales will increase by 5 cones, *ceteris paribus*.

Slopes need not be constant in a given direction, as we have seen in Algebra and Geometry Example 1. Indeed, if the function is nonlinear, then the slopes in various directions depend on the point under consideration. (We could easily imagine that the $C(P, T)$ function earlier was a nonlinear function of price and temperature, for example.) But consider instead another example. Suppose that a firm produces two goods, x and y . We can readily determine cost and revenue associated with any joint output bundle (x, y) . This means that we know profits at any (x, y) production point, since profit is revenue less cost. We can therefore conceptualize a profit function, $\pi(x, y) = \text{Revenue}(x, y) - \text{Cost}(x, y)$. We can also imagine “hiking” on this profit hill in (x, y) space to find the production point associated with the highest profits for the firm. Suppose that we have a bundle (x^*, y^*) that is the top of the profit hill. What do we know about this production point? In particular, what can we say about our feet at this production point? *We can say that our feet are flat.* No matter which direction we turn, our feet should be flat; the slope of the profit function should be zero in all directions. In particular, $\Delta\pi/\Delta x = 0$, and $\Delta\pi/\Delta y = 0$ at (x^*, y^*) . (Were our feet not flat, we would also know which direction to move to get higher on the hill.)

Linear Models

Since we often resort to linear approximations for various curves, it is worthwhile to recall that there are a number of ways to describe a line. Suppose that the line passes through the points $\mathbf{A} = (x_a, y_a)$ and $\mathbf{B} = (x_b, y_b)$. The slope of the line connecting these points is denoted by m , where m equals (using Equation MA.2):

$$m = (y_b - y_a)/(x_b - x_a), \text{ as long as } x_a \neq x_b. \quad (\text{MA.5a})$$

This line can be described using the point-slope formula as:

$$y - y_a = m \cdot (x - x_a). \quad \textit{Point-slope formula} \quad (\text{MA.5b})$$

Point \mathbf{A} could be replaced by point \mathbf{B} in the point-slope formula. If $x_a = x_b$, then the equation of the vertical line is $x = x_a$.

A more visually intuitive method of describing a line uses the slope-intercept formula. This formula is:

$$y = m \cdot x + b. \quad \textit{Slope-intercept formula} \quad (\text{MA.5c})$$

The slope, m , is derived using Equation MA.5a, and intercept, b , represents the value of y when $x = 0$. b is the y intercept, and it can be derived from the point-slope formula in Equation MA.5b by moving y_a to the other side of the equality and regrouping to obtain:

$$b = y_a - m \cdot x_a. \quad \textit{The value of the y intercept } b \quad (\text{MA.5d})$$

One final way to describe a line is using a *parametric representation*. The line connecting \mathbf{A} and \mathbf{B} can be described as the set of points \mathbf{C} where:

$$\mathbf{C} = (x_c, y_c) = (r \cdot x_a + (1 - r) \cdot x_b, r \cdot y_a + (1 - r) \cdot y_b) = r \cdot \mathbf{A} + (1 - r) \cdot \mathbf{B},$$

where $-\infty < r < \infty$. *Parametric representation of a line* **(MA.5e)**

A particularly useful attribute of the parametric representation of a line is that it is easy to describe the line segment between \mathbf{A} and \mathbf{B} by restricting r . If the parameter r is restricted to be between zero and one, then Equation MA.5e describes the *line segment* between \mathbf{A} and \mathbf{B} . In this instance, r tells us how far \mathbf{C} is along the line segment from \mathbf{B} to \mathbf{A} . For example, $r = 3/4$ means that \mathbf{C} is three-fourths of the way from \mathbf{B} to \mathbf{A} , and $r = 0$ means that \mathbf{C} is \mathbf{B} .

When r is restricted to be between zero and one, mathematicians sometimes refer to \mathbf{C} as a convex combination of \mathbf{A} and \mathbf{B} . More generally, \mathbf{C} is a *convex combination* of points, \mathbf{A}_1 – \mathbf{A}_k , if:

$$\mathbf{C} = p_1 \cdot \mathbf{A}_1 + \dots + p_k \cdot \mathbf{A}_k,$$

where $p_i \geq 0$ for $i = 1, \dots, k$, and $p_1 + \dots + p_k = 1$. *Convex combination* **(MA.6)**

The coefficients p_i are restricted to be nonnegative and sum to 1. The coefficient p_i is the weighting factor associated with point A_i . This calculation implies that C is a weighted average of A_i . The weighting factors p_i have a geometric interpretation, just as the parameter r in Equation MA.5e had along the line segment. They can be thought of as describing how much pull the i th point, A_i , has in producing the convex combination C . For example, with $k = 2$ and each weight $p_i = 1/2$, C is midway between the two endpoints. With $k = 3$ and each $p_i = 1/3$, C is the center of the triangle determined by these three points. Each point in the triangle $A_1A_2A_3$ can be described as a convex combination of A_1 , A_2 , and A_3 , given the restrictions $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$ and $p_1 + p_2 + p_3 = 1$. The Excel file for Figure 5.1 provides an example of this situation. This figure shows a three-good budget constraint in which the consumption point shown has $p_1 = 1/2$, $p_2 = 1/3$, and $p_3 = 1/6$. Put another way, half of the individual's income is spent on good 1 (x), one-third is spent on good 2 (y), and one-sixth is spent on good 3 (z).

Linear word problems either provide two points of reference that determine the line or describe the line in terms of slope and intercept. Algebra and Geometry Example 2 shows examples of each type of question, using supply and demand.

Algebra and Geometry Example 2: Working with Slope and Intercept

Facts: No firm will supply output for a price of less than \$1 per unit. From here, firms are willing to supply 20 more units for each \$3 price increase.

Given these facts, provide a graph of supply and provide an equation for supply as a function of price, $X_s(P)$.

Solution: The graph in this case is particularly simple to produce, since the P intercept has been provided. The slope is implicitly provided, since we know that x increases by 20 for every P increase of \$3; the slope of supply is:

$$m = \Delta P / \Delta x = 3/20 = 0.15.$$

Notice that it was not necessary to state that supply was linear; the linearity of supply was implicit in the statement that supply increases by 20 for each \$3 increase in price.

Supply has a P intercept of 1 and a slope of 0.15, so it passes through the point (100, \$16) (over 100 and up 15 from the intercept). Supply is shown in Figure MA.1.

An equation for this line using the slope-intercept formula, Equation MA5c, is:

$$P = 1 + 0.15 \cdot X. \quad \text{Inverse supply} \quad (2a)$$

Note: This is *not* supply (quantity supplied as a function of price), but it is inverse supply.

Equation 2a can be thought of as answering the question: What price is required to support a given level of output X ? To obtain supply, we must solve for X , using Equation 2a. Move 1 to the other side of the equation: $P - 1 = 0.15 \cdot X$. Divide by 0.15 (multiply by $20/3$, since $0.15 = 3/20$) to obtain X as a function of P :

$$X_s(P) = 20/3 \cdot P - 20/3. \quad \text{Supply} \quad (2b)$$

It is worth noting that both equations have implicit restrictions on the values of the independent variable; $X \geq 0$ in Equation 2a, and $P \geq 1$ in Equation 2b.

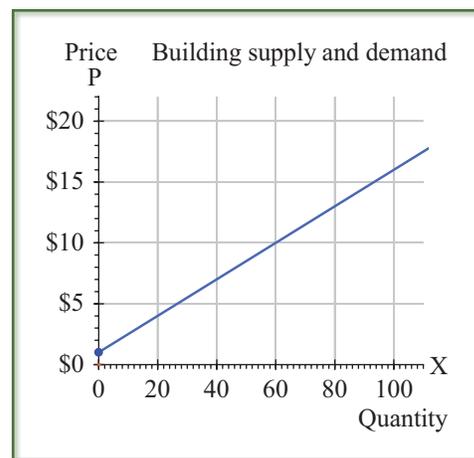


FIGURE MA.1 Graphing Linear Supply

Algebra and Geometry Example 3 uses a word problem in which two points, rather than slope and intercept, are used to determine the line.

Algebra and Geometry Example 3: Working from Two Points

Facts: Demand is linear. At a price of \$14 per unit, 20 units are demanded, but at a price of \$6 per unit, 100 units are demanded.

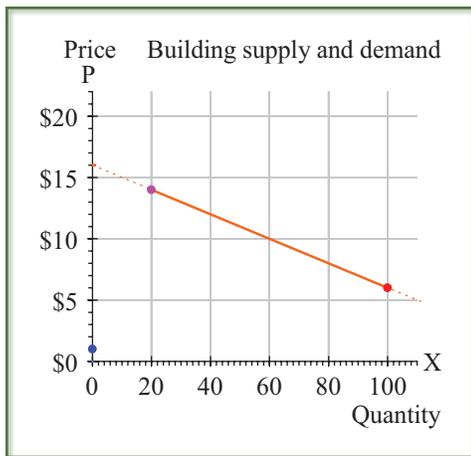


FIGURE MA.2 Graphing Demand from Two Points

Given these facts, provide a graph of demand, and provide an equation for demand as a function of price, $X_d(P)$.

Solution: These points can be plotted on a graph, but the slope-intercept form is not readily apparent. Of course, the intercept can be estimated, given an accurate graph. Such a graph shows a P intercept of \$16. The slope is calculated as $\Delta P/\Delta X = -8/80 = -0.1$. Point-slope form (Equation MA.5a) is: $P - 14 = -0.1 \cdot (X - 20)$. Solving for P, we obtain slope-intercept form of demand:

$$P = 16 - 0.1 \cdot X. \quad \text{Inverse demand} \quad (3a)$$

This is not demand, but rather, inverse demand. To obtain demand, we must solve for X in Equation 3a. Switch P and $-0.1 \cdot X$ (by adding their inverses) to obtain: $0.1 \cdot X = 16 - P$. Multiply by 10 (divide by 0.1) to obtain demand:

$$X_d(P) = 160 - 10 \cdot P. \quad \text{Demand} \quad (3b)$$

This is readily seen as the linear demand curve in Figure MA.2.

We sometimes require discussing quantities using areas. Welfare economics, in particular, uses areas to describe the net benefit from trade, using the notions of consumer surplus and producer surplus. Given linear demand and supply, we can readily calculate such surpluses, using the formulas for the areas of rectangles and triangles.

Geometry facts: *The area of a rectangle that is w wide and h high is $w \cdot h$, and the area of a triangle that is w wide and h high is $w \cdot h/2$.*

We also encounter trapezoids in these calculations. Rather than use a formula for the area of a trapezoid, it is perhaps easier to simply note that a trapezoid can be broken up into triangles and rectangles.

Since economists often rely on linear models of economic phenomena, it is worthwhile to note that, in general, solutions to linear models are readily obtained by solving for one variable in terms of others to reduce the dimensionality of the problem. The strategy to follow is to continue reducing the number of variables in the problem until a final solution is obtained. In general, if we have n equations in n unknowns, we can determine the solution to this problem (i.e., we can find the n unknowns that simultaneously satisfy the n equations). Since this sounds quite abstract, consider the most common two-dimensional example in all of economics: supply and demand.

How can we obtain the market clearing price and quantity in a market, given linear demand and supply? This is a two-dimensional problem; there are two variables to determine. But we can solve this because we have two relationships (equations)—supply and demand. We can approach this by taking either a price or a quantity orientation. The price orientation produces demand and supply as a function of price. The quantity orientation produces inverse demand and inverse supply as a function of quantity. We can work from either orientation to determine market clearing price and quantity.

If we model supply as a function of price, $S(P)$, and demand as a function of price, $D(P)$, then we obtain the equilibrium price by setting supply equal to demand and solving for P. A solution exists because supply is upward sloping and demand is downward sloping; therefore, the lines must intersect. The point of intersection is the market clearing price and quantity. From a mathematical point of view, $S(P) = D(P)$ produces a market clearing price. By substituting this price back into $S(P)$ or $D(P)$, we obtain the common market clearing quantity. Conversely, if we model supply and demand as a function of quantity (and use inverse supply and inverse demand functions), then we have $P_s(X)$ and $P_d(X)$. By setting $P_s(X) = P_d(X)$ and solving for X, we obtain the market clearing quantity. Substituting this quantity back into $P_s(X)$ or $P_d(X)$, we obtain we obtain the common market clearing price.

Algebra and Geometry Example 4 uses the supply and demand derived in Examples 2 and 3 to determine the market clearing price and quantity in this market. It also uses the area of a triangle formula to calculate consumer surplus and producer surplus.

Algebra and Geometry Example 4: Finding an Equilibrium

- Given demand and supply from Equations 3b and 2b, respectively, what is the market clearing price and quantity in this market?
- What is consumer surplus and producer surplus in this market?

Solution:

- $S = D$ when both sides of the market choose the same output. This occurs when $X_d(P) = X_s(P)$. Using Equations 2b and 3b, we have:

$$160 - 10 \cdot P = 20/3 \cdot P - 20/3. \tag{4a}$$

Regrouping:

$$50/3 \cdot P = 500/3; \text{ or } P = \$10. \tag{4b}$$

Quantity is obtained by plugging 10 into S or D from Equations 2b or 3b:

$$X_d(10) = 160 - 10 \cdot 10 = 60 \tag{4c}$$

$$X_s(10) = 20/3 \cdot 10 - 20/3 = 60.$$

Both yield the equilibrium quantity shown in Figure MA.3.

- Consumer surplus is the area beneath the demand curve but above price. This is the triangle bounded by (0, \$16), (0, \$10), and (60, \$10). In this instance, this is a 60-unit-wide by \$6-high triangle. Given this:

$$CS = 60 \cdot \$6/2 = \$180. \tag{4d}$$

Producer surplus is the area above the supply curve but beneath price. This is the triangle bounded by (0, \$1), (0, \$10), and (60, \$10). In this instance, this is a 60-unit-wide by \$9-high triangle. Given this:

$$PS = 60 \cdot \$9/2 = \$270. \tag{4e}$$

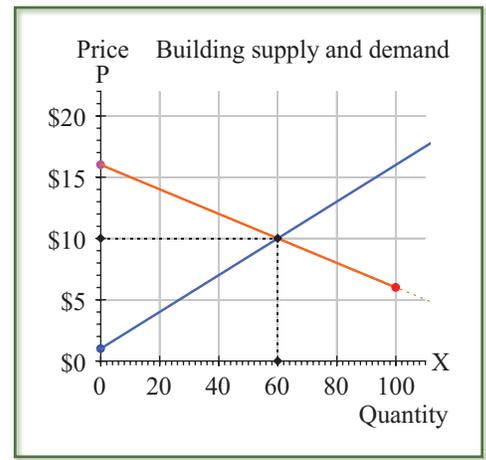


FIGURE MA.3 An Equilibrium between Supply and Demand

Solving Systems of Equations

As noted earlier, you typically will be able to solve a system of equations if you have as many variables as equations. One equation in one unknown, two equations in two unknowns, three equations in three unknowns, and so on generally are all solvable situations. There are many methods for solving such problems, and all will end up with the same solution, as long as the rules of algebra are followed.

The typical method is to reduce the dimensionality of the problem by solving for one of the variables as a function of the others and then substituting this into the other equations. If the system of equations is already solved for a common variable, as in Algebra and Geometry Example 4 with quantity, this happens so quickly (in Equation 4a) that you may not have noticed that you had a system of two equations in two unknowns (supply and demand in Q and P).

The situation is a bit more challenging if both equations are implicit functions of the variables that must be solved for. One ready example comes from solving for monopolistic competition in the long run, discussed in the Chapter 15 Appendix. The following equations and subsequent two sentences are reproduced from that appendix:

$$72 - 2 \cdot m \cdot x = 12 + 0.4 \cdot x \qquad MR(x) = MC(x) \qquad \tag{15A.9a}$$

$$72 - m \cdot x = 1,200/x + 12 + 0.2 \cdot x \qquad p(x) = ATC(x) \qquad \tag{15A.9b}$$

Two equations in two unknowns can, in general, be solved. Solving for x and m , we obtain $x = 40$ and $m = 0.55$. Algebra and Geometry Example 5 shows one way to obtain that solution.

Algebra and Geometry Example 5: Solving Two Equations in Two Unknowns

We must find a method for reducing the dimensionality of the problem. We could solve for x or m , using one of the equations and substituting into the other. Because x is multiplied by m , the algebra involved if this strategy is used is a bit messy.

Another solution is to notice that the structure of both equations has the term $-m \cdot x$. We can solve for $-m \cdot x$ in Equation 15A.9a and substitute that into Equation 15A.9b to remove the m :

$$-2 \cdot m \cdot x = 0.4 \cdot x - 60 \quad (5a)$$

Dividing by 2, we obtain:

$$-m \cdot x = 0.2 \cdot x - 30 \quad (5b)$$

Substituting into Equation 15A.9b, we obtain:

$$72 + 0.2 \cdot x - 30 = 1,200/x + 12 + 0.2 \cdot x \quad (5c)$$

Simplifying:

$$30 = 1,200/x, \text{ or } x = 40 \quad (5d)$$

Substituting Equation 5d into Equation 5b, we obtain:

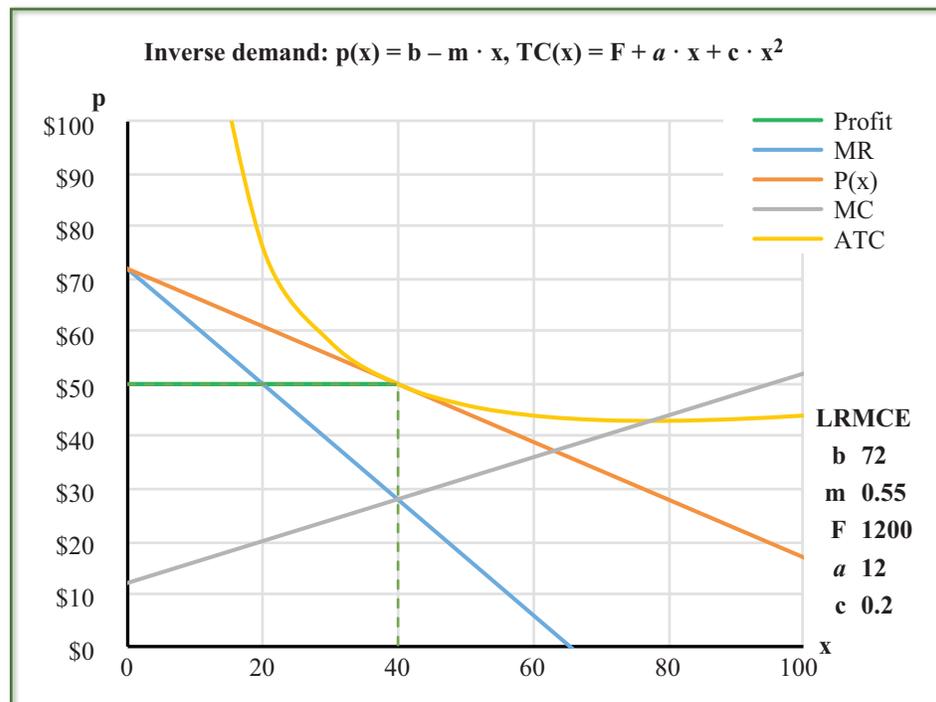
$$-40 \cdot m = 0.2 \cdot 40 - 30 = -22 \quad (5d)$$

Solving for m :

$$m = -22/-40 = 0.55 \quad (5e)$$

Figure MA.4 depicts this solution (this is the same as Figure 15A.1C). $x = 40$ is the point where $MR = MC = 28$, and $P = ATC = 50$, as required in the long run in monopolistically competitive markets. *Note:* We would have obtained the same solution had we solved for $-m \cdot x$ in Equation 15A.9b and substituted that result into Equation 15A.9a.

FIGURE MA.4 Solving for Price and Quantity: Monopolistic Competition in the Long Run



Quadratic Models and the Quadratic Formula

Quadratic functions are often used in microeconomics. Geometrically, the quadratic function $y = a \cdot x^2 + b \cdot x + c$ is a parabola. If $a > 0$, the parabola is a “bowl,” and if $a < 0$, the parabola is a “hill.” The bottom of the bowl or the top of the hill occurs at $x = -b/(2 \cdot a)$, and the parabola is symmetric about this value of x . If the quadratic function is set equal to zero, the equation:

$$a \cdot x^2 + b \cdot x + c = 0 \text{ has solutions:}$$

$$x = \frac{-b \pm (b^2 - 4 \cdot a \cdot c)^{0.5}}{(2 \cdot a)}. \quad \text{Quadratic formula} \quad \text{(MA.7)}$$

The quadratic formula has zero, one, or two solutions, depending on whether $b^2 - 4 \cdot a \cdot c$ is less than zero, equal to zero, or greater than zero.

Exponential and Logarithmic Models

A *logarithm* is an exponent. If $y = \log_A(x)$, then $x = A^y$. Two bases are typically used for A : $A = 10$, and $A = e \approx 2.718$. These two bases have special names: \log_{10} is called *common logarithm*, and \log_e is called *natural logarithm*. Few students find logarithms intuitive, but to the extent that they are, the common logarithm is more intuitive. (For example, if y is a common logarithm, and $y = 1.62$, then we know that x is between 10 and 100 (since $10 = 10^1$, and $100 = 10^2$.) Natural logarithms, most often denoted $\ln(x)$, rather than $\log_e(x)$, are nonetheless more useful for our purposes, and we exclusively use natural logarithms in this text. (Sometimes, you will see the logarithm function written in shorthand without the parentheses as $y = \ln x$.) If $y = \ln(x)$, then $x = e^y$, and x is called the *antilogarithm* of y . It is worth noting that e is exp in Excel, so the equation $=e^y$ is written $=\text{exp}(y)$, rather than $=e^y$. The natural logarithm function in Excel is denoted $\ln(x)$.

It is worthwhile to recall some basic attributes of logarithms and exponents. Although the properties that follow are true for logarithms of any base, base e is used in the current discussion (since this is what we use in this text). Basically, logarithms turn multiplication into addition, and exponentiation into multiplication.

Logarithm rules	$\ln(x \cdot y) = \ln(x) + \ln(y)$.	(MA.8a)
	$\ln(x/y) = \ln(x) - \ln(y)$.	(MA.8b)
	$\ln(x^c) = c \cdot \ln(x)$.	(MA.8c)
Relations between logs and antilogs	$e^{\ln(x)} = x$.	(MA.8d)
	$\ln(e^x) = x$.	(MA.8e)
Power rules for $A > 0$	$A^b \cdot A^c = A^{b+c}$.	(MA.8f)
	$(A^b)^c = A^{b \cdot c}$.	(MA.8g)
	$(A \cdot B)^c = A^c \cdot B^c$.	(MA.8h)
	$1/A^b = A^{-b}$. (A^{-b} is the multiplicative inverse of A^b)	(MA.8i)
	$A^{(1/c)} = \sqrt[c]{A}$. (this is the c th root of A)	(MA.8j)
	$A^b/A^c = A^{b-c}$.	(MA.8k)
	$A^b/A^b = A^{b-b} = A^0 = 1$. (a special case of Equation MA.8k)	(MA.8l)
	$A^{(b/c)} = (A^b)^{(1/c)} = \sqrt[c]{A^b}$	(MA.8m)
	$(A^b)^{(1/b)} = A^1 = A$. (a special case of Equation MA.8m)	(MA.8n)
	Derivative rules	$d(\ln(x))/dx = 1/x$.
$d(e^x)/dx = e^x$.		(MA.8p)

The first three equations show how mathematical operations are transformed by taking logarithms. The next two identities show how logarithms and exponentiation (antilogarithms) “cancel” one another. (Sometimes, identities are shown using a “triple” equal sign, \equiv , rather than the normal “double” equal sign.) The next nine equations provide rules for using exponents. Particular care should be taken to understand negative and fractional exponents, as they are often confused for one another.

Note: *Negative exponents denote powers in the denominator of a fraction, and fractional exponents denote roots (square root is $1/2$, cube root is $1/3$, and so on).*

The last two properties only make sense if you know derivatives. They are shown here so that all of the logarithm and exponentiation rules are in one location.

Logarithms and exponentiation are useful in understanding growth models as well as compounding. Logarithms also provide a useful transformation because they turn multiplicative models (such as the CD utility and the CD production functions, or constant elasticity demand functions) into additive models. Additive models allow for coefficient estimation, using linear regression techniques. Regression analysis is beyond the scope of this text but is an important component of managerial economics texts. (See, for example, Chapter 5 of Keat, Young, and Erfle, *Managerial Economics: Economic Tools for Today's Decision Makers*, 7th edition, Pearson Education, 2013.)

Making Economic Sense of Mathematical Models

One mistake that economics students often make is to forget to check that their mathematical solution makes economic sense. A simple example should suffice to make the point. The constrained optimization problem examined in Chapter 6 asks: How can an individual maximize utility subject to a budget constraint? The answer derived there is that the individual should spend all income and be at a point where $MRS = P_x/P_y$. This ensures that all goods purchased have equal “bang for the buck” (marginal utility per dollar spent). Depending on the utility function chosen, the *mathematical* solution may well lie outside the first quadrant (meaning that either $x < 0$ or $y < 0$) along the budget constraint. But $x < 0$ does not make economic sense. Therefore, if $x < 0$ occurs, the *economic* solution is to buy no x and to spend all income on y (and purchase I/P_y units of y) to maximize utility. Similarly, if $y < 0$ occurs, the *economic* solution is to buy no y and to spend all income on x (and purchase I/P_x units of x) to maximize utility. *Mathematical results should always be tempered by making sure that those results make economic sense.*

Section MA.2 Outline MA.2 A Primer on Differential Calculus

- Derivative Rules
- Using Derivatives for Optimization
- Distinguishing Maximums from Minimums
- Derivatives of Multivariate Functions
- Constrained Optimization

Calculus is not required for this text, but much of the text material indirectly employs calculus because it is based on marginal analysis. The term *marginal* can be recast as the mathematical term *derivative*, so any student who understands marginal analysis also understands (at a practical level) the basics of derivatives.

A typical calculus class examines the theoretical underpinnings of derivatives, using theorems and proofs. Significant time is spent on functions that are of little use in the typical microeconomics class (and on “epsilon delta” proofs that are also of little practical import in the current context). Indeed, from a microeconomic *user's* perspective, differentiation can be boiled down to seven rules, three of which are used regularly.

A derivative is a rate of change, or a slope. Formally, the derivative of a function $f(x)$ is defined as:

$$\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right] = \lim_{\Delta x \rightarrow 0} [\Delta f(x)/\Delta x]. \quad (\text{MA.9})$$

The equation $(f(x + \Delta x) - f(x))/\Delta x$ is simply the slope of the *secant* (or *chord*) from the point $(x, f(x))$ to $(x + \Delta x, f(x + \Delta x))$ (this is also called an *arc slope*). The limit of these arc slopes for smaller and smaller arcs is the derivative at the point.

A derivative requires that the function be smooth at the point under consideration. If the function is not smooth, then the limiting slopes depend on the direction in which Δx approaches zero. For example, there is no derivative at $x = 0$ to the absolute value function $y = |x|$. As noted earlier in the discussion of the term *smooth* in Section MA.1, there are two answers to the slope at $x = 0$ of $y = |x|$, depending on whether Δx approaches zero from below or above. If Δx approaches zero from below, then $x + \Delta x = 0 + \Delta x < 0$, and the slope is -1 . If Δx approaches zero from above, then $x + \Delta x = 0 + \Delta x > 0$, and the slope is $+1$. For a derivative to exist, the same limit must occur, regardless of how Δx approaches zero. The derivative at a point requires that the function be smooth.

There are three shorthand ways to write derivatives. At times, the functional notation denoting x as the variable in $f(x)$ is omitted, and the derivative is written as df/dx . We sometimes see the derivative written as $f'(x)$ or f' ; this is read as *f prime*. And if $y = f(x)$, then the derivative may be written as dy/dx . The letter d in the derivative notation can be thought of as a small “delta,” or incremental change.

The derivative is the limiting rate of change in the function. A derivative is simply a limiting slope. The slope defined in Equation MA.2 is defined over discrete changes in x (between x_1 and x_2). The derivative simply examines what happens to slope as x_1 and x_2 become closer and closer to one another. Put another way, the derivative of $f(x)$ at point x is the slope of the tangent to the curve $f(x)$ at point x .

Derivative Rules

Fortunately, we do not have to use the definition of derivative in Equation MA.9 to *use* calculus because several rules provide us with all that is necessary to apply calculus to a wide variety of microeconomics problems. These rules are provided without proof (for proof, see any calculus text), but the first two are so easy that proofs are readily apparent.

Rule 1: The Constant Rule

$$\text{If } f(x) = a, \text{ then } df(x)/dx = 0. \qquad \text{Constant rule} \qquad \text{(MA.10)}$$

The derivative of a constant is zero. This rule is simple to understand: The slope of a horizontal line is zero. Regardless of the size of the denominator, the numerator of the slope fraction is zero if the function is a constant.

Rule 2: The Line Rule

$$\text{If } f(x) = m \cdot x + b, \text{ then } df(x)/dx = m. \qquad \text{Line rule} \qquad \text{(MA.11)}$$

The derivative of a line is a constant. This, too, is trivial. The defining feature of a line is that the slope is a constant: The calculation of slope as rise over run is the same, regardless of the size of the run and regardless of where on the line this calculation is made.

Rule 3: The Power Rule

$$\text{If } f(x) = a \cdot x^b, \text{ then } df(x)/dx = b \cdot a \cdot x^{(b-1)}. \qquad \text{Power rule} \qquad \text{(MA.12)}$$

In words, the power rule says that you bring down the exponent and multiply it times the function, once the exponent is reduced by one. (If you have not seen calculus, this should not have an intuitive feel. Nonetheless, it is a reasonably straightforward rule to learn.)

Unlike Rules 1 and 2, Rule 3 is *not* immediately obvious. It is also tremendously powerful. The easiest way to visualize this rule is to examine a special case: Set $a = 1$ and consider what happens when $x = 1$ because $f(1) = 1$, regardless of b . Put another way, the point $(x, y) = (1, 1)$ is *always* a point on the curve $y = x^b$; the only distinguishing difference is the slope at this point. And the slope at this point is b , according to Equation MA.12.

Take, for example, the parabola $y = x^2$. The power rule says that the slope of the parabola changes in a linear fashion, according to the rule $dy/dx = 2 \cdot x$. The slope of the parabola at $x = 1$ is 2, and the slope at $x = 1/2$ is 1, for example. You can verify this visually in Figure MA.5A. In each panel in Figure MA.5, the tangent at $(1, 1)$ is provided; the slope at this point is therefore easy to determine, using the grid system.

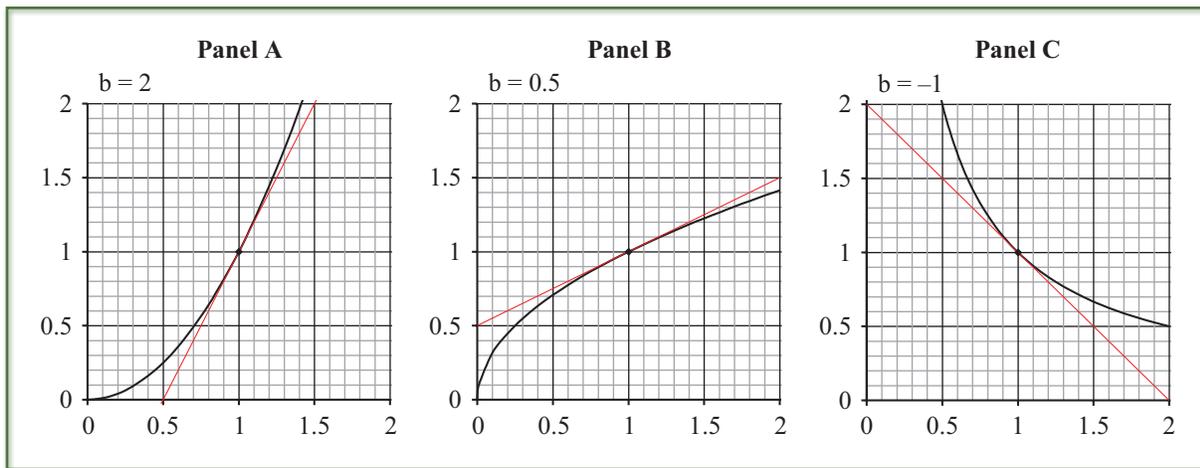


FIGURE MA.5 Examining the Slope of the Power Function $y = x^b$ at $(1, 1)$ for Various Values of b

The slope $dy/dx = b \cdot x^{(b-1)} \equiv b$, given $x = 1$.



The surprising thing about the power rule is that it works regardless of whether b is an integer or not. So, for example, if y is the square-root function, $y = x^{0.5}$, then $dy/dx = 0.5 \cdot x^{-0.5}$. At $x = 1$, the slope is $1/2$, and at $x = 1/4$, the slope is 1 , as can be seen in Figure MA.5B. Finally, b need not be positive. As a result, if $y = x^{-1}$, then $dy/dx = -x^{-2}$. At $x = 1$, the slope is -1 , and at $x = 1/2$, the slope is -4 , as can be seen in Figure MA.5C. The Excel version of this file allows you to vary b between -2 and $+4$ in 0.1 increments.

Rule 4: The Sum Rule

If $f(x) = g(x) + h(x)$, then $df(x)/dx = dg(x)/dx + dh(x)/dx$. *Sum rule* (MA.13)

This can be stated verbally as: The derivative of a sum of functions is the sum of the derivatives of those functions.

The sum rule simply allows you to decompose the derivative of a function of a number of terms into a series of derivatives. This allows you, for example, to take the derivative of a polynomial by treating each term in turn. For example, if $y = x^3 + x^2 + x + 1$, then $dy/dx = 3 \cdot x^2 + 2 \cdot x + 1$ by repeated application of the sum rule, together with the power rule.

Some authors provide a separate rule for differences, but since $a - b = a + (-b)$, it is not actually necessary to have a separate rule for the difference between two functions. Indeed, some authors simply call this the sum/difference rule.

Calculus Example 1: MR = MC Is an Application of the Sum Rule

The most important rule in producer theory (to maximize profits, a firm must set $MR = MC$) is simply an application of the sum rule (this rule is the focus of Chapter 12). Since profits are total revenue less total cost,

$$\pi(x) = TR(x) - TC(x). \quad (1a)$$

To maximize profits, we must get to the top of the profit hill. This occurs when $d\pi/dx = 0$.

Mathematical hint: Q: How do you know that you are at the top of a smooth hill?

A: Your feet are flat.

But:

$$d\pi/dx = dTR(x)/dx - dTC(x)/dx, \text{ according to the sum rule.} \quad (1b)$$

Since marginal is the derivative of total, $dTR(x)/dx = MR(x)$, and $dTC(x)/dx = MC(x)$, so:

$$d\pi/dx = 0 \text{ implies that } MR(x) - MC(x) = 0, \text{ or that } MR = MC. \quad (1c)$$

The sum rule and the power rule are the workhorses for many applications in microeconomics. Many problems can be solved using these rules, and these rules actually contain the first two rules as special cases. (If $b = 0$, then $x^b = 1$, and the power function becomes $f(x) = a \cdot x^b = a$. The power rule implies that the slope is zero, since $df(x)/dx = b \cdot a \cdot x^{(b-1)} = 0$, given $b = 0$. But this is simply the first rule. And the second rule is obtained by breaking up the linear function $m \cdot x + b$ into $m \cdot x$ and b , using the sum rule, and noting that $d(m \cdot x)/dx = m$ and $db/dx = 0$, according to the power rule.)

Calculus Example 2: Solving a Profit-Maximization Problem

Suppose that you face demand for x given by:

$$x(P) = 100 - 2 \cdot P. \quad \text{Demand} \quad (2a)$$

Your total cost function is:

$$TC(x) = 100 + 20 \cdot x + x^2. \quad \text{Total cost} \quad (2b)$$

You wish to maximize profits. How much output should you produce, and what profits will you earn?

Note: We can solve this problem in two ways. One solution involves deriving the profit function and then maximizing profits; the other involves obtaining marginal revenue and marginal cost functions. Both lead to the same solution, and both employ the sum and power rules. We show only the first method here.

Profits are total revenue less total cost. We have total cost, but we need total revenue. Total revenue must be a function of quantity, so we need to obtain price as a function of quantity (i.e., we need inverse demand). We obtain inverse demand by solving for P in Equation 2a:

$$P(x) = 50 - x/2. \quad \text{Inverse demand} \quad (2c)$$

Multiply inverse demand by quantity to obtain total revenue as a function of quantity:

$$TR(x) = P(x) \cdot x = 50 \cdot x - x^2/2. \quad \text{Total revenue} \quad (2d)$$

Profits are simply total revenue less total cost, so we have:

$$\pi(x) = TR(x) - TC(x) = -1.5 \cdot x^2 + 30 \cdot x - 100. \quad \text{Profit function} \quad (2e)$$

(It is worth pointing out that some students will misuse algebra and calculate profits as: $\pi(x) = TR(x) - TC(x) = 0.5 \cdot x^2 + 70 \cdot x - 100$. These students fail to distribute the minus sign to all three terms in the TC function.)

Profits are maximized when the top of the profit hill is achieved. This occurs when the profit hill is flat. This means that a slope of zero, $d\pi/dx = 0$, is required for profit maximization (as noted in Calculus Example 1). This derivative can be easily obtained by applying the sum rule to Equation 2e:

$$d\pi(x)/dx = d(-1.5 \cdot x^2)/dx + d(30 \cdot x)/dx - d(100)/dx \quad (2f)$$

and the power rule to Equation 2f:

$$d\pi(x)/dx = -3 \cdot x + 30. \quad \text{Derivative of profits} \quad (2g)$$

Setting this equal to zero produces $x = 10$. Substituting $x = 10$ into the profit function in Equation 2e produces a profit-maximizing profit level of $\pi(10) = 50$.

Rule 5: The Product Rule

$$\begin{aligned} &\text{If } f(x) = g(x) \cdot h(x), \\ &\text{then } df(x)/dx = g(x) \cdot dh(x)/dx + h(x) \cdot dg(x)/dx. \end{aligned} \quad \text{Product rule} \quad (\text{MA.14})$$

This is stated verbally as: The derivative of a product of two functions is the first function times the derivative of the second, plus the second function times the derivative of the

first. (The order is unimportant; the point is that each part of the product must be multiplied by the derivative of the other part.)

Until you get used to using the product rule, it is worthwhile to write out the functions g and h , as well as their derivatives. If written in “matrix” fashion, as done in Equations 3a and 3b, the derivative is obtained by summing the “cross-product” of terms.

Calculus Example 3: Applying the Product Rule

Suppose that $y(x) = x^2 \cdot (x^{1/2} + 1)$. What is dy/dx ?

$$\text{Let:} \quad g(x) = x^2, \quad \text{and} \quad h(x) = x^{1/2} + 1. \quad (3a)$$

$$\text{In this event:} \quad dg(x)/dx = 2 \cdot x, \quad \text{and} \quad dh(x)/dx = 1/2 \cdot x^{-1/2}. \quad (3b)$$

The product rule states that:

$$dy/dx = g(x) \cdot dh/dx + h(x) \cdot dg/dx. \quad (3c)$$

Substituting (“cross” of Equations 3a and 3b):

$$= x^2 \cdot (1/2 \cdot x^{-1/2}) + (x^{1/2} + 1) \cdot 2 \cdot x. \quad (3d)$$

Simplifying:

$$= 5/2 \cdot x^{3/2} + 2 \cdot x. \quad (3e)$$

This function was chosen so that it would be easy to independently verify the answer derived from the product rule. (Expanding the function, we obtain $y(x) = x^{5/2} + x^2$. This function is easily differentiated, using the sum and power rules, to obtain Equation 3e.) If the product is more complex, this would be less obvious. This is the value of the product rule.

Some texts describe a separate rule for the derivative of a quotient, but the rule is complex and rarely used in microeconomics. As a result, it is worthwhile to understand that the quotient rule need not be used because it can always be turned into an application of the joint use of the product rule, the power rule, and the chain rule (as we show in Calculus Example 6).

Rule 6: The Chain Rule (also known as “the function of a function rule”)

$$\text{If } f(x) = g(h(x)), \text{ then } df(x)/dx = (d(g(h))/dh) \cdot h(x)/dx. \quad \textit{Chain rule} \quad (\text{MA.15})$$

The chain rule is like unpeeling an onion, layer by layer. The derivative of a function, which is itself a function, is taken one step at a time. This rule tends to cause significant confusion; therefore, it is worthwhile to write out the functions explicitly until you feel comfortable enough with the rule to do it as second nature. When attacking a chain rule problem, it is typically easiest to work from the inside of the onion out—first define $h(x)$, and then define $g(h)$, rather than moving from g to h .

Calculus Example 4: Applying the Chain Rule

Suppose that $f(x) = (x^2 + 1)^2$. What is $df(x)/dx$?

$$\text{Let:} \quad g(h) = h^2, \quad \text{and} \quad h(x) = x^2 + 1. \quad (4a)$$

$$\text{In this event:} \quad dg(h)/dh = 2 \cdot h, \quad \text{and} \quad dh(x)/dx = 2 \cdot x. \quad (4b)$$

The chain rule states:

$$df(x)/dx = dg/dh \cdot dh/dx. \quad (4c)$$

$$= 2 \cdot h \cdot 2 \cdot x. \quad (4d)$$

$$= 2 \cdot (x^2 + 1) \cdot 2 \cdot x = 4 \cdot x^3 + 4 \cdot x. \quad (4e)$$

As with Calculus Example 3, this can be verified by expanding the square to obtain $f(x) = x^4 + 2 \cdot x^2 + 1$ and then differentiating to obtain Equation 4e.

Calculus Example 5 expands on Calculus Example 4 by adding another link to the chain rule chain.

Calculus Example 5: Adding Links to the Chain

The only difference from Calculus Example 4 is that the x in the x^2 part of $f(x)$ is replaced by a function $k(x)$ (rather than x).

Suppose that $y = ((2 \cdot x^{0.5} - 1)^2 + 1)^2$. What is dy/dx ?

$$\text{Let:} \quad g(h) = h^2, \quad h(k) = k^2 + 1, \quad \text{and} \quad k(x) = 2 \cdot x^{0.5} - 1. \quad (5a)$$

$$\text{In this event:} \quad dg(h)/dh = 2 \cdot h, \quad dh(k)/dk = 2 \cdot k, \quad \text{and} \quad dk(x)/dx = x^{-0.5}. \quad (5b)$$

The chain rule states:

$$dy/dx = dg/dh \cdot dh/dk \cdot dk/dx. \quad (5c)$$

$$= 2 \cdot h \cdot 2 \cdot k \cdot x^{-0.5}. \quad (5d)$$

$$= 2 \cdot (k^2 + 1) \cdot 2 \cdot (2 \cdot x^{0.5} - 1) \cdot x^{-0.5}. \quad (5e)$$

$$= 4 \cdot ((2 \cdot x^{0.5} - 1)^2 + 1) \cdot (2 \cdot x^{0.5} - 1) \cdot x^{-0.5}. \quad (5f)$$

As mentioned earlier, the chain rule is like unpeeling an onion: Take the derivative of each layer and continue doing so until the derivative with respect to x is taken. Calculus Example 5 shows three layers; therefore, the derivative has three parts: dg/dh , dh/dk , and dk/dx .

Calculus Example 6: The Quotient Rule

The quotient rule is not worth memorizing, since it is rarely used in economics and, as shown in this example, can be derived by applying the chain rule, the power rule, and the product rule.

Suppose that $f(x) = g(x)/h(x)$. What is df/dx ? Let $k(x) = k(h(x)) = (h(x))^{-1}$. Put another way:

$$k(h) = h^{-1}. \quad (6a)$$

Given this, the power rule implies:

$$dk/dh = -h^{-2}. \quad (6b)$$

The chain rule requires:

$$dk/dx = -h^{-2} \cdot dh(x)/dx \quad (6c)$$

We can rewrite the quotient as a product:

$$f(x) = g(x) \cdot k(h(x)). \quad (6d)$$

The product rule states:

$$df/dx = g(x) \cdot dk(x)/dx + k(x) \cdot dg(x)/dx. \quad (6e)$$

Substituting Equation 6c into Equation 6e:

$$df/dx = g(x) \cdot (-h^{-2} \cdot dh(x)/dx) + (h(x))^{-1} \cdot dg(x)/dx. \quad (6f)$$

Regrouping, we obtain:

$$df/dx = (h(x) \cdot dg(x)/dx - g(x) \cdot dh(x)/dx)/(h(x))^2. \quad (6g)$$

Equation 6g is formally known as the quotient rule.

The preceding six rules (MA10–MA15) are all that is necessary to take most derivatives that are encountered in microeconomics. Rule 7, the final rule, involves special

functions that are sometimes used in microeconomics and that cannot be attacked using the previous six rules.

Rule 7: Special Functions

The natural logarithm and exponential function are sometimes encountered in microeconomics, and as a result, it is worth laying out the derivatives of these functions.

$$d(\ln(x))/dx = 1/x. \quad \text{Derivative of natural logarithm (MA.16a)}$$

$$d(\ln(u(x)))/dx = (1/u(x)) \cdot d(u(x))/dx. \quad \text{Natural log + Chain rule (MA.16b)}$$

$$d(e^x)/dx = e^x. \quad \text{Derivative of the exponential function (MA.17a)}$$

$$d(e^{u(x)})/dx = e^{u(x)} \cdot d(u(x))/dx. \quad \text{Exponential function + Chain rule (MA.17b)}$$

Equations MA.16a and MA.17a are the simple forms, and Equations MA.16b and MA.17b are provided to remind you that the chain rule is often used in conjunction with the exponential or logarithmic rule because the exponential or logarithmic function is often a more complex function of x than the identity function ($u(x) = x$ is called the identity function).

Using Derivatives for Optimization

Microeconomics is based on marginal analysis, and the word *marginal* means slope. As we noted earlier, derivatives are slopes, so derivatives are used throughout microeconomics. One of the main uses of derivatives is to find extreme points—points where maximums and minimums occur. Calculus Examples 1 and 2 examine how profits are maximized. Calculus Example 1 derives the $MR = MC$ rule as an application of the sum rule, and Calculus Example 2 uses a specific demand and cost function to derive the profit-maximizing output level. Both are based on a simple notion:

Fact: *If a smooth function has a maximum or a minimum at a point, then the derivative at that point must be zero.*

The reason why slope must be zero is simple: Consider once again the profit function. If this is not true, then the sign of the slope tells a direction to change to increase profits. Two possibilities exist:

1. If $d\pi/dx > 0$ at output level x_0 , then increasing output will increase profits.
2. If $d\pi/dx < 0$ at output level x_0 , then decreasing output will increase profits.

These are the possibilities depicted graphically in Figures 12.14A and 12.14B.

The same can be said of minimums: If you wish to find the minimum of a function $f(x)$, you must find the point where $df/dx = 0$. Consider Calculus Example 7.

Calculus Example 7: Cost Minimization

Suppose that you face equal-weighted CD production of:

$$Q(L, K) = (L \cdot K)^{0.5}. \quad \text{CD production function (7a)}$$

Suppose that you wish to produce 4 units of output at minimum cost, given $P_L = 1$ and $P_K = 4$. (Questions such as this form the basis for Chapter 10.) Which input bundle (L, K) produces 4 units of output at minimum cost? To answer this question, we need to create an input cost function:

$$C(L, K) = P_L \cdot L + P_K \cdot K = L + 4 \cdot K. \quad \text{Input cost function (7b)}$$

This input cost function describes the cost at any (L, K) bundle, but not all bundles produce 4 units of output. We can restrict the bundles to those that produce 4 units of output by solving for either variable as a function of the other in Equation 7a, given $Q = 4$. The isoquant $Q = 4$ is:

$$4 = (L \cdot K)^{0.5}. \quad \text{Q = 4 isoquant (7c)}$$

Solving for K , we obtain:

$$K = K(L) = 16/L. \quad K(L) \text{ points along } Q = 4 \text{ isoquant} \quad (7d)$$

If we substitute $K(L)$ in place of K in Equation 7b, we obtain the cost of producing 4 units of output as a function of L :

$$C(L) = L + 64/L. \quad \text{Cost of producing } Q = 4 \text{ as a function of } L \quad (7e)$$

To minimize the cost of producing 4 units of output, we take the derivative of $C(L)$ and set it equal to zero:

$$dC/dL = 1 - 64/L^2 = 0. \quad \text{Slope of } C(L) \text{ set equal to zero} \quad (7f)$$

Solving for L , we obtain $L = 8$. Substituting $L = 8$ into Equation 7d, we obtain $K = 2$ (and we note that, as expected, the input bundle $(L, K) = (8, 2)$ produces $Q = 4$, given the production function in Equation 7a. Substituting $L = 8$ back into $C(L)$ yields a cost of 16 to produce 4 units of output at minimum cost.

This problem is an example of the *substitution method* discussed in the Chapter 6 Appendix. The objective function is the cost function (Equation 7b), and the constraint is the isoquant (Equation 7c). The constrained optimization problem is turned into an unconstrained optimization problem by folding the constraint into the objective function (as is done in Equation 7e). This problem can also be solved using the Lagrangian method, as we will see in Calculus Example 13. We return to a discussion of constrained optimization after we discuss multivariate derivatives.

Distinguishing Maximums from Minimums

From a geometric perspective, maximums and minimums have the same derivative condition: $df/dx = 0$. If you are out hiking, and you are at the top of a hill or at the bottom of a valley, the same condition holds: Your feet are flat. The condition $df/dx = 0$ is called a *first-order condition*. Mathematicians say that the first-order condition is *necessary but not sufficient* for the optimum. To distinguish between maximums and minimums requires examining the *second-order condition* at the critical output level (a critical output level is one where the first-order condition is satisfied).

Consider walking along in the x direction. If x^* is a *critical output level*, we know that $df(x^*)/dx = 0$. If this is the bottom of a valley, for x just below x^* , we would have $df/dx < 0$, and for x just above x^* , we would have $df/dx > 0$. This implies that df/dx is *increasing* at the point where $df/dx = 0$. Put another way, the slope of $df/dx > 0$ at this output level. But the slope of df/dx is simply the *second derivative* of f with respect to x . A second derivative is written as d^2f/dx^2 . (Note that the notation puts the superscript 2s in different locations; a second derivative is *not* the square of the first derivative.) Sometimes, a second derivative is denoted using two prime signs: f'' is an alternative way to denote d^2f/dx^2 . The second-order test for a minimum can be summarized as:

x^* is a local minimum of the function $f(x)$ if:

$$df(x)/dx = 0 \text{ at } x = x^*, \text{ and } d^2f/dx^2 > 0 \text{ at } x = x^*. \quad \text{Local minimum test} \quad (\text{MA.18})$$

Calculus Example 8: Distinguishing Maximums from Minimums

Find the critical value of:

$$f(x) = x^2 - 6 \cdot x + 15. \quad (8a)$$

Determine if this is a maximum or a minimum. Provide a geometric description of this function.

$$df/dx = 2 \cdot x - 6. \quad (8b)$$

Setting the derivative equal to zero produces a critical x value of $x = 3$. Plugging this back into the objective function produces $y = f(3) = 6$. The point $(x, y) = (3, 6)$ is the bottom of a parabola. (You may recall that quadratic functions are parabolas. The parabola is a “bowl” when there is a positive coefficient on the quadratic term and a “hill” when there is a negative coefficient on the quadratic term.) This is confirmed by the second derivative test:

$$d^2f/dx^2 = 2 > 0. \quad (8c)$$

We can use Equation MA.18 to verify that $L = 8$ is a cost-minimizing solution to Calculus Example 7. Equation 7f stated that $dC/dL = 1 - 64/L^2$. Given this, $d^2C/dL^2 = 128/L^3$, using the power rule. Evaluating d^2C/dL^2 at $L = 8$ yields $d^2C/dL^2 = 128/8^3 = 1/4 > 0$; therefore, $L = 8$ is, indeed, a minimum, according to Equation MA.18.

Conversely, if the critical output level x^* is at the top of a hill, then for x just below x^* , we would have $df/dx > 0$, and for x just above x^* , we would have $df/dx < 0$. This implies that df/dx is *decreasing* at the point where $df/dx = 0$. Put another way, the slope of $df/dx < 0$ at $x = x^*$, or $d^2f/dx^2 < 0$ at $x = x^*$. The second-order test for a maximum can be summarized as:

x^* is a local maximum of the function $f(x)$ if:

$$df(x)/dx = 0 \text{ at } x = x^*, \text{ and } d^2f/dx^2 < 0 \text{ at } x = x^*. \quad \text{Local maximum test} \quad (MA.19)$$

Calculus Example 9: Finding the Top of a Hill

This example starts with the total revenue hill, given linear demand described in Equation 2d: $TR(x) = 50 \cdot x - x^2/2$. Rather than maximize profits, we wish to determine what value of x maximizes total revenue. This occurs when $dTR/dx = 0$.

$$dTR/dx = 50 - x = 0. \quad (9a)$$

The top of the total revenue hill occurs at $x = 50$. At this output level, $TR(50) = 1,250$. The point $(x, y) = (50, 1,250)$ is the top of a parabola, and this fact is confirmed by the second derivative test:

$$d^2TR/dx^2 = -1 < 0. \quad (9b)$$

There is a single critical value in each of Calculus Examples 8 and 9. This need not be the case; more than one critical value may exist. In this event, the second derivative test in Equations MA.18 and MA.19 can distinguish between maximums and minimums. Calculus Example 10 examines this issue.

Calculus Example 10: Profit Maximization or Profit Minimization?

Suppose that total cost is given by:

$$TC(x) = 1,000 + 900 \cdot x - 25 \cdot x^2 + x^3/3. \quad (10a)$$

The firm can sell as little or as much as it wishes at a price of $P_x = \$500$ per unit of x . How many units of x should the firm produce to maximize profits? Before determining this value of x , consider the following question:

Q10: In what type of market structure is this firm embedded?*

Profits are $TR - TC$, so we can write profits as:

$$\pi(x) = TR(x) - TC(x) = 500 \cdot x - (1,000 + 900 \cdot x - 25 \cdot x^2 + x^3/3).$$

$$\pi(x) = -1,000 - 400 \cdot x + 25 \cdot x^2 - x^3/3. \quad (10b)$$

*Q10 answer: The firm is embedded in a perfectly competitive market because it can sell as much or as little as it wishes without affecting the market-determined price of \$500 per unit. Questions such as this are examined in Chapter 13.

Profits are maximized when $d\pi/dx = 0$. The derivative of profits is:

$$d\pi/dx = -400 + 50 \cdot x - x^2. \quad (10c)$$

The profit-maximizing level of x therefore satisfies:

$$0 = -400 + 50 \cdot x - x^2. \quad (10d)$$

This can be solved using the quadratic formula, Equation MA.7, or by factoring. In this instance, factoring is straightforward:

$$0 = -(x - 10) \cdot (x - 40). \quad (10e)$$

This implies that $x = 10$ and $x = 40$ are critical values of the profit function. To distinguish maximums from minimums, we take the second derivative and check the sign of the second derivative at $x = 10$ and $x = 40$ (according to Equations MA.18 and MA.19). The second derivative is the derivative of Equation 10c:

$$d^2\pi/dx^2 = 50 - 2 \cdot x. \quad (10f)$$

At $x = 10$, $\pi'' = 30 > 0$, so that $x = 10$ is a profit minimum, according to Equation MA.18. By contrast, at $x = 40$, $\pi'' = -30 < 0$, so that $x = 40$ is a profit maximum, according to Equation MA.19. This is verified by plugging $x = 10$ and $x = 40$ back into the profit function in Equation 10b: $\pi(10) = -\$2,833.33$, and $\pi(40) = \$1,666.67$. We should finally note that it would be better to shut down than produce at $x = 10$: $\pi(0) = -\$1,000$.

Note: This example is qualitatively the same as Figure 13.5, which shows a profit minimum at $x = 3$ and a profit maximum at $x = 17$.

Derivatives of Multivariate Functions

The derivative rules laid out earlier in this section apply to multivariate functions as well as univariate functions. There are a number of multivariate derivative concepts, but the most important is the notion of a partial derivative.

Partial Derivatives

A partial derivative is a derivative with respect to a single variable, holding all other variables fixed. This is the notion of *ceteris paribus* in mathematical form. All variables save one are held fixed, and the derivative is taken in a single direction. If the function has n independent variables, then there are n partial derivatives. If $f(\mathbf{X}) = f(x_1, x_2, \dots, x_n)$, then the derivative in the i th direction is denoted as $\partial f/\partial x_i$, or as f_i , where $i = 1, \dots, n$. The symbol ∂ is the *del* operator, and $\partial f/\partial x_i$ is read as “the partial derivative of f in the i th direction,” or “the partial of f with respect to x_i .”

The most difficult thing about taking a partial derivative is to treat the variables that are constant as constants. For example, if $f(x, y) = x^2 \cdot y + 2 \cdot y$, then $\partial f/\partial x = 2 \cdot x \cdot y$, and $\partial f/\partial y = x^2 + 2$. Each partial treats the other variable as a constant.

The easiest way to visualize a partial derivative is to imagine that you are trying to climb a hill. Think of the height of the hill as a function. The coordinates on the hill can be described using two axes—an east/west axis (denoted x) and a north/south axis (denoted y). If you are hiking on a hill, you can always describe the slope in the x direction (by pointing your feet to the east) and in the y direction (by pointing your feet to the north). Both slopes make analytical sense, and both slopes are partial derivatives. You know that you are on the top of a hill if, regardless of the direction your feet are pointed, they are flat. This allows us to solve multivariate profit-maximization problems, as shown in Calculus Examples 11 and 12.

Calculus Example 11: Finding the Top of a Multivariate Profit Hill, Take 1

Suppose that profits are a function of price, P , and advertising, a , according to the function:

$$\pi(P, a) = 600 \cdot P - 100 \cdot P^2 + 5 \cdot P \cdot a - 0.25 \cdot a^2 - 200. \quad (11a)$$

What are the values of P and a that will maximize profits?

The top of the profit hill occurs when both partial derivatives equal zero. The partial derivatives are:

$$\pi_p = 600 - 200 \cdot P + 5 \cdot a. \quad \text{Partial derivative with respect to price} \quad (11b)$$

$$\pi_a = 5 \cdot P - 0.5 \cdot a. \quad \text{Partial derivative with respect to advertising} \quad (11c)$$

Both partial derivatives will equal zero at the top of the profit hill, so we must set each partial equal to zero. This provides two equations in two unknowns. Setting $\pi_a = 0$ and solving for a , we obtain:

$$a = 10 \cdot P. \quad (11d)$$

Substituting this into $\pi_p = 0$ produces:

$$600 = 150 \cdot P. \quad (11e)$$

Solving for P , we obtain:

$$P = 4 \text{ and } a = 40. \quad \text{Profit-maximizing price and advertising levels} \quad (11f)$$

To determine profits in this instance, substitute $(4, 40)$ into profit Equation 11a:

$$\pi(4, 40) = 2,400 - 1,600 + 800 - 400 - 200 = 1,000. \quad \pi_{max} \text{ profits} \quad (11g)$$

The second multivariate example considers a firm that sells two goods. How much of each good should be produced to maximize profits?

Calculus Example 12: Finding the Top of a Multivariate Profit Hill, Take 2

Suppose that a firm produces two goods, x and y , and finds that profits are a function of the quantity of each produced, according to the function:

$$\pi(x, y) = 20 \cdot x - x^2 + 16 \cdot y - 2 \cdot y^2 - 2 \cdot x \cdot y. \quad (12a)$$

What are the values of x and y that will maximize profits?

The top of the profit hill occurs when both partial derivatives equal zero. The partial derivatives are:

$$\pi_x = 20 - 2 \cdot x - 2 \cdot y. \quad \text{Partial derivative with respect to } x \quad (12b)$$

$$\pi_y = 16 - 4 \cdot y - 2 \cdot x. \quad \text{Partial derivative with respect to } y \quad (12c)$$

Both partial derivatives will equal zero at the top of the profit hill, so we must set each partial equal to zero. This provides two equations in two unknowns. Setting $\pi_x = 0$ and solving for y , we obtain:

$$y = 10 - x. \quad (12d)$$

Substituting this into $\pi_y = 0$ produces:

$$0 = 16 - 4 \cdot (10 - x) - 2 \cdot x. \quad (12e)$$

Solving for x , we obtain:

$$x = 12. \quad \text{Top of the profit hill—level of } x \text{ production} \quad (12f)$$

Substituting $x = 12$ into Equation 12d produces the optimal y of $y = -2$. Profits in this instance are $\pi(12, -2) = 104$. *Although this is the top of the mathematical hill, this answer makes no economic sense.* As noted at the end of Section MA.1, all mathematical results should be checked to make sure they make economic sense: We must temper this mathematical result with economic reality. If the top of the hill has a negative level of y , then the best *economic* outcome will have $y = 0$. If we restrict $y = 0$ in Equation 12a, we have:

$$\pi(x; y = 0) = 20 \cdot x - x^2. \quad (12g)$$

The partial derivative in the x direction is the same as Equation 12b with $y = 0$. This produces a top of the economic hill at $x = 10$. *Note in particular that x is not 12, as derived in Equation 12f.* The top of the restricted hill (i.e., with $y = 0$) need not occur at the same level of x as the unrestricted hill.

Profits in this instance are $\pi(10, 0) = 100$. The economic solution is less than the top of the mathematical hill, but it represents the best the firm can do in this instance.

Gradients

Given a function $f(\mathbf{X})$, the *gradient of f* is defined as:

$$\nabla f = (\partial f / \partial x_1, \dots, \partial f / \partial x_n). \quad \text{Gradient of } f \quad \text{(MA.20)}$$

A gradient is a vector of partial derivatives. The gradient of a function is a vector that points in the direction of maximum rate of change. Imagine that you are hiking on a hill; you can always tell which direction is the steepest ascent by noting in which direction your feet are pointed upward the most steeply. At different points on the hill, you will have different directions of maximal change.

The MU vector, MP vector, and price vector are all examples of gradients used in the text. Each of these vectors is perpendicular to the level set of the function at the point under consideration. Again, it is easiest to imagine hiking. If you are pointed in the direction of maximum ascent, then the quickest descent is behind you, and the direction of no ascent or decline (walking flat) is to your left and right. This, of course, assumes that the hills you are walking on are smooth (i.e., differentiable). In economic terms, the MU vector is perpendicular to the indifference curve, the MP vector is perpendicular to the isoquant, and the price vector is perpendicular to the budget constraint (or isocost line).

Total Derivatives

Sometimes, independent variables are not truly independent of one another. There may be a functional relation that exists between two independent variables. In this instance, a total derivative describes the change in the function f , taking into account not only the direct effect of the variable that has changed but also the effect of that change on other variables in the system. For example, the equation for a total derivative with respect to x_1 is:

$$df(\mathbf{X})/dx_1 = (\partial f / \partial x_1) + (\partial f / \partial x_2) \cdot (dx_2/dx_1) + \dots + (\partial f / \partial x_n) \cdot (dx_n/dx_1). \quad \text{(MA.21)}$$

The first component is the direct effect due to a change in x_1 ; the second is the indirect effect caused by the change in x_2 caused by the change in x_1 , and so on. A similar total derivative could be defined with respect to each of the n independent variables. (We should note that the total derivative makes sense as an application of the chain rule.)

Calculus Example 13: The Total Effect of a Price Change

We can use total derivatives to examine why demand might be backward bending in the used-car market (see the Lemons model in Section 21.2). Demand is a function of price, P , and average quality, A : $x = f(P, A)$ in this market. We would expect $f_P < 0$ and $f_A > 0$. But average quality is likely to be an increasing function of price in this market, $dA/dP > 0$. As a result, an increase in price has competing effects (this is the derivative version of Equation 21.3b):

$$\begin{aligned} dx/dP &= \partial f / \partial P + [\partial f / \partial A \cdot dA/dP]. & \text{(13a)} \\ ? &= - + [+ \cdot +]. \end{aligned}$$

The first term on the right-hand side is the pure price effect on demand; the second bracketed term is the secondary effect on demand due to an increase in average quality caused by an increase in price. The net sign of an increase in price on demand is indeterminate in this setting.

Constrained Optimization

Many problems have the following form:

Maximize an objective function subject to one or more constraints.

The objective need not be to maximize; it could be to minimize. Therefore, we often call the problem a constrained optimization problem, rather than a constrained maximization or minimization problem.

In consumer theory, for example, the objective is to maximize utility, subject to being able to afford the bundle under consideration. The utility-maximizing consumption bundle is the best the consumer can do, subject to the prices and income faced by that consumer. Equation 6.1 describes the solution to this problem, and the Chapter 6 Appendix provides two calculus-based methods to solve that problem. One method is called the *substitution method*, and the other is called the *Lagrangian method*. Since that appendix lays out these two methods and provides a comparison of these methods in the context of maximization, we focus here on the same comparison in the context of a minimization problem.

The Substitution Method

The substitution method reduces the dimensionality of the problem by introducing the constraint into the objective function. Calculus Example 7 uses this method. The objective is to minimize the objective function (input cost), subject to a production constraint. The objective function is a function of two variables, L and K: $C(L, K) = L + 4 \cdot K$, according to Equation 7b. The constraint function is that 4 units of output must be produced. In mathematical terms, the (L, K) input bundle must satisfy the isoquant $Q_0 = 4 = (L \cdot K)^{0.5}$, according to Equation 7c. The substitution method next solves for one of the variables, using the constraint function (Equation 7d), and *substitutes* the result into the objective function. This substitution reduces the two-dimensional constrained optimization problem into a one-dimensional unconstrained function of labor in Equation 7e.

The Lagrangian Method

The Lagrangian method increases the dimensionality of the problem by introducing the constraint into the objective function. This strategy sounds like it would make the problem harder, but in many instances, it makes it easier.

The Lagrangian method requires that the constraint function be written in a form that equals zero. This is then multiplied by the *Lagrange multiplier* (typically the Greek letter lambda, λ) and added to the objective function to obtain the *Lagrangian function*, \mathcal{L} . The Lagrangian is a function of the variables from the objective function, \mathbf{X} , together with the Lagrange multiplier, λ (or Lagrange multipliers, if there are multiple constraint functions under consideration). If the objective function $f(\mathbf{X})$ is subject to a constraint function, $g(\mathbf{X}) = g_0$. The Lagrangian function is:

$$\mathcal{L}(\mathbf{X}, \lambda) = f(\mathbf{X}) + \lambda \cdot (g_0 - g(\mathbf{X})). \quad \text{A Lagrangian function} \quad (\text{MA.22})$$

The Lagrangian is a function of $n + 1$ variables. The optimum of a function occurs at the point where the derivative is zero. With multiple variables, multiple partial derivatives must simultaneously equal zero. By taking the partial derivative of the Lagrangian with respect to each of the $n + 1$ variables and setting them equal to zero, we obtain the maximum of the objective function subject to the constraint. At one level, this sounds surprising, since \mathcal{L} is not the objective function f . But at the optimum point, \mathcal{L} is the objective function f ! The reason why this is true is straightforward: The optimum occurs when $\partial \mathcal{L} / \partial \lambda = 0$, but that restriction is simply the constraint function. If the constraint function equals zero, the Lagrangian is the objective function in this instance.

We already know the solution to the minimization problem in Calculus Example 7 solved via the substitution method, but it is instructive to see this same problem solved by Lagrangians.

Calculus Example 14: Cost Minimization via Lagrangians

Minimize $C(L, K) = L + 4 \cdot K$, subject to $4 = (L \cdot K)^{0.5}$, using Lagrangians.

$$\mathcal{L}(L, K, \lambda) = L + 4 \cdot K + \lambda \cdot (4 - (L \cdot K)^{0.5}) \quad (14a)$$

The three first-order conditions are therefore:

$$\partial \mathcal{L} / \partial L = 0, \quad \text{or} \quad 1 - \lambda \cdot 0.5 \cdot (K/L)^{0.5} = 0. \quad (14b)$$

$$\partial \mathcal{L} / \partial K = 0, \quad \text{or} \quad 4 - \lambda \cdot 0.5 \cdot (L/K)^{0.5} = 0. \quad (14c)$$

$$\partial \mathcal{L} / \partial \lambda = 0, \quad \text{or} \quad 4 - (L \cdot K)^{0.5} = 0, \text{ or } 4 = (L \cdot K)^{0.5}. \quad (14d)$$

By bringing the second term to the other side of the equation in Equations 14b and 14c, we obtain:

$$1 = \lambda \cdot 0.5 \cdot (K/L)^{0.5}. \quad (14e)$$

$$4 = \lambda \cdot 0.5 \cdot (L/K)^{0.5}. \quad (14f)$$

Dividing Equation 14e by Equation 14f, we obtain:

$$1/4 = K/L, \text{ or } L = 4 \cdot K. \quad (14g)$$

Substituting $L = 4 \cdot K$ into Equation 14d, we obtain:

$$4 = ((4 \cdot K) \cdot K)^{0.5} = 2 \cdot K, \text{ or } K = 2. \quad (14h)$$

Given $K = 2$ and Equation 14g, we obtain $L = 8$. This is the same solution found in Calculus Example 7, using the substitution method. This should not be surprising because the Lagrangian's first-order conditions provide exactly the same information as the substitution method. Both produce the dual requirement for constrained optimum: The constraint function must be satisfied, and $MRTS = P_L/P_K$ (the same requirements described in Equation 10.5).

MA.3 Using Excel in Economic Modeling

Excel was used extensively to create this text, but for the most part, you have not been asked to create economic models using Excel. The Excel figure files rely on sliders and click boxes, rather than on you entering data and equations. But Excel is also a powerful tool that *you* can use for your own economic modeling, as discussed in Section 7.5 and also in Chapter 19. Chapter 19, in particular, was written with the idea that it could be taught very early in the course, as a way of teaching how to use Excel for economic modeling. The discussion in Section 19.2 describes using Excel to solve present value problems. This discussion assumes no knowledge of Excel but makes no attempt to be exhaustive in its treatment of Excel's functionalities. The present discussion follows the same general outline but provides a few more details via the use of small examples, rather than a more unified discussion of a specific topic such as the Section 19.2 discussion of present value or the Section 7.5 discussion of choice among constraints. In the discussion in this section, a **boldface** word signifies an Excel command or name. It is worth having Excel open as you read this material.

Excel's General Layout

When you open Excel, you will note one or more ribbons at the top of the screen and a series of tabs at the bottom of the screen. Excel 2007 (and later versions) shows the **Home** ribbon, together with other ribbons (**Insert**, **Page Layout**, etc.). (If using Excel 2003, you see toolbars, including the **Standard** and **Formatting** toolbars, together with the line that shows the pull-down menus [**File**, **Edit**, **View**, etc.].) To determine what a specific icon on a ribbon/toolbar does, simply move the mouse cursor over the icon. Many of the icons carry over from Word and are self-explanatory. The tabs at the bottom of the

Section MA.3 Outline

- Excel's General Layout
- Entering Equations
- Dragging Information
- Relative versus Absolute Referencing
- Structuring Your Worksheet
- Using Excel's Goal Seek Function

screen (labeled **Sheet1**, **Sheet2**, **Sheet3**) list available worksheets in the workbook. The highlighted worksheet is the one currently being used. The main part of the worksheet is a spreadsheet in which each cell has two references: The letter is a column reference, and the number is a row reference. The upper left corner is cell A1. Just above the spreadsheet is the **Formula Bar** (a white space starting just above the C and D columns next to the f_x symbol [= sign in Excel 2003]).

Entering Equations

If you wish to type an equation in a cell, simply click on that cell and type in the equation. (Clicking on a cell involves placing the mouse over the cell and then left-clicking once. Most simple operations involve left-clicks; when we say click, we mean left-click.) For example, to multiply two times three in cell A1, click on A1 and then type $=2*3$ (↵ signifies pressing the Enter key). Underscored words, numbers, or parts of an equation in this section signify that these should be entered from the keyboard (or via mouse clicks for cell entry) to follow the Excel example. Once the Enter key is pressed, the number 6 will appear in cell A1.

Multiplying two times three does not showcase any of Excel's power. One of the things Excel does very well is repetitive calculation. Therefore, we turn to an example that highlights this ability. In the process, you will learn about dragging equations.

Dragging Information

To harness Excel's power to do repetitive operations, you must learn how to get Excel to enter information in cells (rather than having to do it yourself). If you begin a pattern, Excel will do its best to continue that pattern. Perhaps the best example of this is that you can quickly get the months of the year entered in successive cells. Do the following: First type Jan↵ in a cell. Next, **drag** that cell in whichever direction you wish. To do this, follow these steps.

Excel rule: *Do the following to **drag** a cell: (1) Click on that cell; (2) move the mouse toward the lower right corner of the cell until you see a black cross; (3) depress the mouse click and drag either sideways or vertically.*

This produces a listing of months with three-letter abbreviations. (If you did not get this, you depressed the mouse when crossed arrows were showing; crossed arrows move cells.)

If you wish to have full names of the months, simply start with January rather than Jan. This example shows how information in one cell can be dragged to another cell. The same is true of equations. You can also drag more than one cell at a time, as we will see a little later.

Excel Example 1: Dragging an Equation

Suppose that you wish to know the value of a function for a series of values of x . In particular, suppose that we wish to answer the following question:

For how many numbers x between 1 and 10 is the function $f(x)$ an integer, given:

$$f(x) = 1 + 2 \cdot (4 + x^2)/10? \quad (1a)$$

Excel can easily answer this question. (This equation was not chosen because it has economic content [it does not] but because the answer is not immediately obvious, and solving this question readily highlights some of Excel's power.)

Click on cell A1 (put the mouse over A1 and then left-click once) and type 1 ↵. The number 1 should show up in cell A1 and cell A2 should be *active*. (The active cell is seen in two ways: First, a black border surrounds this cell, and second, the active cell is listed in the **Name Box** [to the left of the **Formula Bar** above the Column A label].) Notice that typing ↵ moves you down one cell. Now type:

$$=1+2*(4+A1^2)/10 \quad (1b)$$

TABLE MA.2 The Solution to Example 1 Obtained via Excel

1	2	3	4	5	6	7	8	9	10
2	2.6	3.6	5	6.8	9	11.6	14.6	18	21.8

to enter the equation in cell A2. The number 2 should show up in cell A2, and cell A3 should be highlighted.

This equation was chosen to show that order of operations works just as discussed in Section MA.1: Parentheses have the highest priority; within the parentheses, the exponent is done first (^ is how exponents are denoted in Excel), then the addition inside the parentheses, followed by the multiplication and division, followed by the outside addition. The equation was also chosen to show that numbers can be entered in an indirect fashion in an equation by referencing a cell that contains a number (in this instance, the number 1 is entered in place of x in the equation in cell A2 by referencing cell A1).

Note that the number 2, rather than an equation, appears in cell A2. To see what is actually in cell A2, simply move back to that cell (this is easily done by using the up arrow [\uparrow] on the keyboard or by moving the mouse to that cell and clicking on the cell). If you do this, Equation 1b will appear in the **Formula Bar**.

At this point, we know that $f(1) = 2$. To find other values, we can click on cell A1 and enter other numbers, *or* we can use Excel's built-in capabilities to do repetitive functions by dragging A1:A2 across the screen. (A colon between two cell references means that the block of cells whose corners are the listed cells are under consideration.) To drag a group of cells in a row or a column, move the mouse's *open white cross* over one of the endpoint cells (for example, cell A1). Next, hold down the mouse's click button, and with the white open cross showing, move to the other endpoint (cell A2). Once you release the click button, the group of cells will be highlighted. If you move the mouse's open white cross to the lower right corner of cell A2, the white cross will first turn into *crossed black arrows* (which allows you to move the cells) and then a *black cross* (which allows you to drag the cells). Once you see the black cross, depress and hold the click button down and drag the pair of cells sideways. As you do, numbers will appear according to how many copies have been created. Stop when you see 10 (this will be in Column J). Table MA.2 shows what will appear in cells A1:J2. We see that four numbers between 1 and 10 produce integer answers (1, 4, 6, and 9).

Excel has provided the answer via the entry of a *single* equation that has been dragged. Row 1 contains values of x , and Row 2 contains values of $f(x)$. It is worth clicking on cell J2 to see that the equation no longer contains a reference to cell A1—it now contains a reference to J1. (And if you check other equations in Row 2, you will see that the equation references the cell just above it to find the data required to perform the calculation.) This is called *relative referencing*. The basic calculations have been quickly derived—and without the keystroke errors that sometimes occur when an equation is repeatedly entered into a calculator).

It is worth noting that Excel *assumed* that you wanted to increase the numbers in Row 1. Excel has built-in routines that try to guess and fill in when patterns are expected. These guesses are not always correct, but they are often helpful. (You can read more about this in Section 19.2A, “Creating Patterns.”)

We could, of course, clean up Table MA.2 by more effectively formatting the data and adding labels. We generally ignore formatting issues in this appendix; you can read about this topic using the **Microsoft Excel Help** information (from the question mark in the upper right corner [next to the close and minimization icons] in Excel 2007, or on the **Standard** toolbar, the **Format** pull-down menu in Excel 2003).

Relative versus Absolute Referencing

When equations are written in Excel, it is often worthwhile to fix some parts of the equation but let other parts vary as the equation is dragged. Equation 1b shows how *relative*

referencing works. As the equation is moved, the new equation looks for data in a location that has the same relative orientation as the original equation. In the case of Equation 1b, the equation that referenced cell A1 was located in cell A2. If this equation is moved anywhere, it will look for data in the cell directly above it. (If you try to copy this equation to the first row, the result is #REF! #REF! is an error code that acknowledges that you cannot reference something in Row 0.)

Sometimes, you want to write an equation that refers to data that do not move. In this event, use *absolute referencing*.

Excel rule: To fix a column, place a dollar sign (\$) in front of the letter reference; to fix a row, place a dollar sign in front of the number reference. If both row and column are fixed, then this is called absolute referencing.

Suppose that you wish to know how much a dollar is worth if it is received at various times in the future. Specifically, suppose that you wish to solve Excel Example 2.

Excel Example 2: A Repetitive PV Question

What is the present value of \$1, received 1, 2, 3, or 4 years from today, given an annual discount rate of 10%?

The present value of X dollars, received t years from now, given discount rate r, is given by Equation 19.1a, reproduced here:

$$PV(X) = X/(1 + r)^t \quad (2a)$$

This equation can be entered into Excel in a variety of ways. Suppose that the discount rate is placed in cell A1. This is accomplished by typing 0.1 in cell A1. (Notice that 10% does not need to be entered; the decimal version is sufficient, and you do not need to format the cell.) Enter time in cells A2:D2 by entering the numbers 1 through 4 in those cells. (These can be directly entered, or the numbers 1 and 2 can be entered [to establish a pattern] and then the pattern established in A2:B2 can be dragged to D2, as described earlier. Interestingly, the increment needs to be entered in this instance; if you try to drag the number 1 from A2 alone, the number remains 1.)

The five numbers in cells A1 and A2:D2 are the “data” for the calculations required to answer Excel Example 2. From here, a single equation produces the desired result, once that equation is dragged. The Excel version of Equation 2a (with X = 1), entered in cell A3, is:

$$=1/(1+\$A\$1)^A2 \quad (2b)$$

The \$A\$1 in the equation signifies that, as the equation is moved, the cell reference remains cell A1.

Once the equation in A3 is dragged to D3, the results are as shown in Table MA.3. The labeling shown in cells B1:C1 and E2:E3 in Table MA.3 is not necessary, but is provided to remind you what is in each cell. (Once again, the numbers are not nicely formatted, but they provide worthwhile information very quickly.) As Equation 2b in cell A3 is dragged sideways, the reference to the discount rate (in A1) does not change (because it is an absolute reference), but the reference to time (in A2:D2) does (because it is a relative reference). This is confirmed by clicking on cell D3, which shows the equation:

$$=1/(1+\$A\$1)^D2 \quad (2c)$$

Finally, we should note what these results suggest. If you are slated to receive \$10,000 4 years from today, and your discount rate is 10%, then you would be indifferent between that option and receiving \$6,830.13 today.

TABLE MA.3 A One-Way PV Table

0.1	r	Note: The equation =1/(1+\$A\$1)^A2 is entered in cell A3.			
1	2	3	4	t = Time	
0.909091	0.826446	0.751315	0.683013	Value of \$1 received t years from today	

As noted earlier, if you are writing an equation in Excel and you want a column reference to remain fixed, use a dollar sign (\$) in front of the letter. If you are writing an equation and you want a row reference to remain fixed, use a dollar sign in front of the row number. If you want both to remain fixed, use a dollar sign in front of each part (as we did earlier with \$A\$1). Equation 2b uses both absolute and relative referencing, but two other possibilities are apparent. The other two possibilities involve *mixed referencing*. Excel Example 3 shows how you might use mixed referencing in equations to answer questions.

Excel Example 3: Creating a Two-Way PV Table

Suppose that you wish to know how the information in Table MA.3 varies as discount rate varies. In particular, suppose that you wish to see the value of $1/(1+r)^t$ for various values of r and t (for example, for $t = 1, 2, \dots, 4$, and for $r = 0.02, 0.04, \dots, 0.1$). This kind of table is often found as an appendix in accounting, economics, and finance texts, and you can create it with one equation.

One solution is to go back to Excel Example 2 and replace the contents of cell A1 with 0.02 (2%) and then copy the answers, replace with 0.04 (4%) and then copy the answers, and so on. But it is less work to use mixed referencing instead. Start with a new worksheet. You can add worksheets to your workbook by clicking the \oplus symbol next to the sheet tab at the bottom of the screen (in Excel 2003, click **Insert, Worksheet** on the **Insert** pull-down menu). To maintain the look started earlier (with years across columns), put years in cells B1:E1, and put the five discount rates in A2:A6. Click on cell B2 (the first entry in the table you are “creating”) and enter the equation:

$$=1/(1+\$A2)^{B\$1} \quad (3a)$$

We place a dollar sign in front of the A in A2 because the equation requires that the discount rate remain fixed as we drag the equation sideways. But the discount rate should change as we drag the equation down; therefore, omit the dollar sign in front of the 2. Put another way, *we want the discount rate to be absolute for columns but relative for rows*. Therefore, we put a dollar sign in front of only the column reference (A in \$A2).

The time variable (B1) is exactly the opposite. We wish to have time vary as we drag the equation across columns, but it should not change as we drag down across rows. *We want time to be relative for columns but absolute for rows*. Therefore, we put a dollar sign in front of only the row reference (1 in B\$1). This equation can then be dragged horizontally and vertically to fill in the cells from B2:E6. The result is Table MA.4. All entries in B2:E6 have been generated from a single dragged equation. Notice that the last row in this table is Table MA.3.

Rather than worry about the equation and the referencing all at once, it is useful to first enter the equation without dollar signs and then go back and add the dollar signs to individual cell references as required.

Excel hint: *To minimize equation entry mistakes, enter the equations you wish to drag in two steps: First, enter the equation with relative referencing (no dollar signs); then add dollar signs as necessary.*

TABLE MA.4 The Value of \$1 t Years from Today Based on Alternative Discount Rates r

	1	2	3	4	$t = \text{Time}$
0.02	0.980392	0.961169	0.942322	0.923845	
0.04	0.961538	0.924556	0.888996	0.854804	
0.06	0.943396	0.889996	0.839619	0.792094	
0.08	0.925926	0.857339	0.793832	0.73503	
0.1	0.909091	0.826446	0.751315	0.683013	
$r = \text{Discount rate}$	The equation $=1/(1+\$A2)^{B\$1}$ is in cell B2.				

Equations with cell references can be entered by clicking on cells, rather than by typing in the cell reference (i.e., if you are typing an equation and you click on cell A2, the reference to A2 appears in the equation). If you click on cells, the reference shows up as relative—that is, there are no dollar signs. If you click reference cells A2 and B1 as you enter the equation, you will obtain:

$$=1/(1+A2)^{B1} \leftarrow \quad (3b)$$

Once an equation such as Equation 3b has been initially entered, click on the cell containing the equation (in the present example, this is in cell B2). Then go to the **Formula Bar** and click on a cell reference you wish to alter. (For example, click anywhere near A2 in the **Formula Bar**.) From here, you can directly enter a dollar sign by appropriately moving the cursor in front of the A and then entering a dollar sign. You can also add dollar signs via the F4 key.

The dollar signs that fix column and row are used so much in Excel that Excel has a special function key devoted to this purpose.

Excel hint: *The F4 key rotates through the four possible referencing alternatives (\$A\$2, A\$2, \$A2, A2, \$A\$2, etc.).*

Do the same for each part of the equation that requires other than relative referencing, and you then have an equation that is ready to drag both horizontally and vertically.

If you are still confused about absolute versus relative referencing, then it may be worthwhile to look at Table 19.5. This table shows all four forms of referencing based on a simple matrix of numbers.

Structuring Your Worksheet

Since Excel is excellent at repetitive calculations, you should carefully consider how you set up your calculations to take advantage of this ability. The creation of Table MA.4 is a case in point. By placing one variable across columns (time) and the other across rows (interest rate), we created a two-way table using absolute and relative referencing.

A more common situation involves a series of calculations based on assumed data. When a model is created from assumed data, it is natural to perform a sensitivity analysis (or comparative statics analysis) on that data. What happens to the net present value of an investment if the discount rate is 6% rather than 5%, for instance? It is worthwhile to have a general structure for your worksheet in which the data are located together in a single area at the top of the worksheet and the calculations based on that data are presented underneath. The discussion of present value problems in Chapter 19 shows many examples of this basic architecture. One of the reasons for doing this is that it is always easy to return to cell A1 (even if A1 is not currently visible on the screen).

Excel hint: *By pressing Ctrl and Home at the same time, you will jump to cell A1.*

When performing calculations, it is *imperative* to cell-reference data, rather than type the data into the equation. A common mistake students make is typing in numbers in the calculating area of the model rather than cell-referencing. If 0.1 is entered in a present value equation because that is the assumed discount rate, then if you wish to perform a comparative statics analysis on the discount rate, you would have to find each time that 0.1 was entered in an equation and change it to a new value. A much more reliable strategy is to cell-reference the assumed discount rate in the equation (as we did in Equations 2b and 3b). If this is done, then all that needs to happen to perform a comparative statics analysis is to change the discount rate in the data area at the top of the worksheet and see how that alters the calculations beneath.

When entering data and setting up the analysis, take time to label cells so that comparative statics analysis is readily accomplished. *Make sure that the cell labels are in the cell next to the data, rather than in the cell with the data.* If you type in 10% discount

rate \leftarrow in a cell and then reference that cell in an equation, you will get an error message (#VALUE!) because Excel no longer considers the cell numeric.

Using Excel's Goal Seek Function

One particularly powerful tool in the Excel toolkit is a built-in function called **Goal Seek**. **Goal Seek** allows you to adjust an equation's value by changing one of the parameters within the equation. This is useful for breakeven analysis, for finding the internal rate of return, or for doing a variety of comparative statics problems. **Goal Seek** is discussed in the context of complex problems involving choice among constraints in Section 7.5 and present value problems in Section 19.3. Excel Example 4 is a much more modest problem involving investments.

Excel Example 4: Answering Questions with Goal Seek

Suppose that you invest \$10,000 at 6% today and maintain that investment for 10 years. How much will you have in 10 years? Given this information, the answer is easily calculated, using the future value formula (Equation 19.1b):

$$FV = 10,000 \cdot (1.06)^{10} = \$17,908.48. \quad (4a)$$

Suppose that you are disappointed with this result and wonder how much higher a return you would have had to invest at to achieve \$20,000 after 10 years. This is easy to solve analytically, but it is also easy to solve using **Goal Seek**. (The analytical answer is simply $r = 2^{0.1} - 1 = 0.071773$, which is obtained by solving for r in the equation: $20,000 = 10,000 \cdot (1 + r)^{10}$.) The (middle of deriving the) **Goal Seek** solution is shown in Table MA.5. This is a screenshot taken after Step 6 in the list of steps that follows (so that the pop-up menu is still visible).

These steps are based on the simple setup where the three pieces of data (initial investment amount, interest rate, and time) are entered into A1:A3 (and labeled in B1:B3). Once the data have been added to A1:A3, the future value formula is placed in cell A4:

$$=A1*(1+A2)^{A3} \leftarrow. \quad (4b)$$

Not surprisingly, given Equation 4a, the result shown in cell A4 is 17908.48. The question is: How much higher must the annual percentage rate (APR) be before the value in A4 is 20,000? To answer this question, follow the steps involved in performing a **Goal Seek** (all cell references are to Table MA.5):

1. Click on the cell you wish to change. In this instance, that is cell A4.
2. Click **Data, What If Analysis, Goal Seek** in Excel 2007 (or **Tools, Goal Seek** in Excel 2003). The **Goal Seek** menu will pop up (shown in D1:G9).
3. Click on the **To value** part of the **Goal Seek** pop-up menu.

TABLE MA.5 A Screenshot after Step 6 of the Goal Seek Process (showing a completely filled-in Goal Seek pop-up menu)

	A	B	C	D	E	F	G
1	10000	Amount now					
2	0.06	APR					
3	10	t = Time					
4	17908.48	Value in time t					
5							
6							
7							
8							
9							

4. Enter the number you wish to have this cell equal. In this instance, that number is 20000.
5. Click on the **By changing** cell part of the **Goal Seek** pop-up menu. Three possibilities exist in this instance: You could change the initial amount (in A1), the APR (in A2), or time (in A3).
6. Click on the cell you wish to change—Table MA.5 shows that A2 was chosen. (Note that dollar signs are automatically added to the clicked cell reference.) You could alternatively type A2 (or even a2) in the **By changing** area.
7. Click **OK**. **Goal Seek** will then provide a status report (i.e., it will tell you if it has found a solution).
8. If it has found a solution, click **OK** to see the solution.

Once **OK** is clicked twice, the result is $APR = 0.071773$, seen in cell A2. You would require approximately a 7.2% return on your investment to double your money in 10 years.
