As submitted to J Schiff, 20/3/96

REARRANGING THE DAVENPORT FORMULA

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September 1995

ABSTRACT. The Summability by Primes formula is a variant of the Arithmetic Fourier Transform. We prove that the formula converges when applied to a large class of even periodic step functions. A special case is proved, using an interesting connection to a formula of Euler.

1. INTRODUCTION

Let f be an even integrable function of period 2π, normalised so that

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = 0.$$

Then if f satisfies certain conditions[1], we can calculate its Fourier cosine coefficients a_n using the *Arithmetic Fourier Transform*:

$$\text{(AFT)} \qquad a_n = \sum_{k=1}^{\infty} \mu(k) S(nk).$$

Here μ is the Mobius function defined on the natural numbers by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is not square-free,} \\ (-1)^q & \text{if } n \text{ is square-free and has } q \text{ (distinct) prime factors} \end{cases}$$

and $S(n)$ is the average value of f sampled at n points given by

$$S(n) = \frac{1}{n} \sum_{j=0}^{n-1} f(2\pi j/n).$$

[1]For example, if $f \in \mathrm{Lip}_\alpha[0, 2\pi]$ where $1/2 < \alpha \leq 1$, or if f is of bounded variation and in $\mathrm{Lip}_\alpha[0, 2\pi]$ for any α. Walker's proof of this will soon appear in [1].

Walker [4] developed a variant of the AFT known as *Summability by Primes*, and showed that it converges to the correct value in some cases for which the AFT does not[2]. To state the Summability by Primes formula, we need the notation

$$\delta_k^q = \begin{cases} 0 & \text{if } k \text{ has a prime factor larger than the } q\text{th prime,} \\ 1 & \text{otherwise.} \end{cases}$$

Then for suitable f, we have

$$a_n = \lim_{q\to\infty} \sum_{k=1}^{\infty} \delta_k^q \, \mu(k) S(nk). \tag{SBP}$$

Schiff and Walker [3] showed that (AFT) holds when f is a step function, but it was not previously known whether the same is true of the Summability by Primes formula. In the next section we show that (SBP) does indeed hold for a large class of step functions.

2. Rearranging the Davenport formula

Davenport [2] proved a result that Schiff and Walker later termed the *Davenport formula*:

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} \{k\theta\} = -\frac{1}{\pi} \sin 2\pi\theta.$$

Here $\{\cdot\}$ is the *first Bernoullian function* given by

$$\{t\} = \begin{cases} 0 & \text{if } t \text{ is an integer,} \\ t - [t] - \frac{1}{2} & \text{otherwise.} \end{cases}$$

The Davenport formula can be used to prove results on the Arithmetic Fourier Transform, but to prove analagous results for the Summability by Primes formula, we need something slightly different. Therefore the first objective of this section is to prove that the *rearranged Davenport formula* holds for almost all θ:

$$\lim_{q\to\infty} \sum_{k=1}^{\infty} \delta_k^q \frac{\mu(k)}{k} \{k\theta\} = -\frac{1}{\pi} \sin 2\pi\theta \tag{RDF}$$

Another way of writing this will be more convenient. Reorder the square-free numbers according to the greatest prime factor they contain: 1, 2, 3, 6, 5, 10, 15, 30, 7, 14,... and call the kth number in this sequence s_k. For each natural number N, let t_N denote the greatest number such that s_{t_N} has no prime factor greater than N. Actually, t_N is always a power of 2 but we will not use this fact. A crucial property of the t_N that

[2]Specifically, Walker showed that the Summability by Primes formula is valid whenever the Fourier series of f is absolutely convergent; this is not the case for the standard Arithmetic Fourier Transform

we *will* use is that $\{1, 2, \ldots N\} \subset \{s_1, s_2, \ldots s_{t_N}\}$. With the new notation, we can write (RDF) as

$$\text{(RDF}'\text{)} \qquad \lim_{N\to\infty} \sum_{k=1}^{t_N} \frac{\mu(s_k)}{s_k}\{s_k\theta\} = -\frac{1}{\pi}\sin 2\pi\theta.$$

Define

$$T_N(\theta) = \sum_{k=1}^{t_N} \frac{\mu(s_k)}{s_k}\{s_k\theta\} + \frac{1}{\pi}\sin 2\pi\theta.$$

(In words, $T_N(\theta)$ is just the difference between the two sides of (RDF′) if the sum on the left is truncated once all s_k with factors no greater than N have been included).

Lemma 1. *Fix $\epsilon > 0$. Then as $N \to \infty$,*

$$\int_0^1 T_N(\theta)^2 d\theta = O\left(\frac{1}{N^{1-\epsilon}}\right).$$

Proof. According to Davenport, it is well-known that

$$\int_0^1 \{m\theta\}\{n\theta\}\, d\theta = \frac{1}{12}\frac{(m,n)^2}{mn},$$

and

$$\int_0^1 \{m\theta\} \sin 2\pi\theta\, d\theta = \begin{cases} -\frac{1}{2\pi} & \text{if } m = 1, \\ 0 & \text{if } m > 1. \end{cases}$$

Thus

$$(*) \qquad \int_0^1 T_N(\theta)^2\, d\theta = \frac{1}{12}\sum_{m,n\le t_N} \frac{\mu(s_m)\mu(s_n)(s_m,s_n)^2}{{s_m}^2{s_n}^2} - \frac{1}{\pi^2} + \frac{1}{2\pi^2}.$$

But from Davenport we also have that

$$\sum_{m,n=1}^{\infty} \frac{\mu(s_m)\mu(s_n)(s_m,s_n)^2}{{s_m}^2{s_n}^2} = \frac{6}{\pi^2}.$$

So we can replace the $\sum_{m,n\le t_N}$ in $(*)$ with $\frac{6}{\pi^2} - \sum_{m,n>t_N}$, obtaining

$$\int_0^1 T_N(\theta)^2\, d\theta = \frac{1}{12}\left(\frac{6}{\pi^2} - \sum_{m,n>t_N} \frac{\mu(s_m)\mu(s_n)(s_m,s_n)^2}{{s_m}^2{s_n}^2}\right) - \frac{1}{2\pi^2}$$

$$= -\frac{1}{12}\sum_{m,n>t_N} \frac{\mu(s_m)\mu(s_n)(s_m,s_n)^2}{{s_m}^2{s_n}^2}.$$

So

$$\left|\int_0^1 T_N(\theta)^2\, d\theta\right| \le \frac{1}{12}\sum_{m,n>t_N} \frac{(s_m,s_n)^2}{{s_m}^2{s_n}^2}.$$

But as remarked earlier, $\{1,2,\ldots N\} \subset \{s_1, s_2, \ldots s_{t_N}\}$, so by including some extra terms we conclude

$$\begin{aligned}
\left|\int_0^1 T_N(\theta)^2\, d\theta\right| &\leq \frac{1}{12}\sum_{m,n>N}\frac{(m,n)^2}{m^2n^2} \\
&\leq \frac{1}{12}\sum_{m>N}\sum_{n=1}^{\infty}\frac{(m,n)^2}{m^2n^2} \\
&= \frac{1}{12}\sum_{m>N}\frac{1}{m^2}\sum_{d|m}\sum_{\substack{n' \text{ with} \\ (n',m)=1}}\frac{1}{n'^2} \\
&\leq \frac{1}{12}\sum_{m>N}\frac{1}{m^2}\sum_{d|m}\sum_{n'=1}^{\infty}\frac{1}{n'^2} \\
&= O\left(\sum_{m>N}\frac{d(m)}{m^2}\right),
\end{aligned}$$

where $d(m)$ is the number of divisors of m. But a result proved on page 260 of Hardy and Wright [5] tells us $d(m) = O(m^\epsilon)$ for any fixed $\epsilon > 0$. Applying this and estimating the resulting sum with an integral gives

$$\left|\int_0^1 T_N(\theta)^2\, d\theta\right| = O\left(\sum_{m>N} m^{\epsilon-2}\right) = O\left(N^{\epsilon-1}\right),$$

proving the lemma. □

Before going on we need to establish the following lemma.

Lemma 2. *Let r be a positive integer, and S the set*

$$\left\{\begin{matrix} n \in \mathbb{N} \,|\, n \text{ is square-free, } n \text{ has at least one prime} \\ \text{factor} > r^2, \text{ and } n \text{ has no prime factor} \geq (r+1)^2 \end{matrix}\right\}$$

Then the sum of reciprocals of the elements of S is $O\left(\frac{\log r}{r}\right)$, as $r \to \infty$.

Proof. For a set A of positive integers, we will denote the sum of the reciprocals of the elements of A by $\sigma(A)$. As usual, p_q is the qth prime. Suppose the primes less than r^2 are $p_1, p_2, \ldots p_k$, and the primes between r^2 and $(r+1)^2$ are $p_{k+1}, \ldots p_{k+l}$. Let U be the set of square-free integers whose prime factors are a subset of $\{p_{k+1}, \ldots p_{k+l}\}$. Let U_m be the subset of U consisting of numbers with m (distinct) prime factors. Then U is the disjoint union of the U_m, $m = 1, 2, \ldots l$. But every element of U_m is at least as big as $(p_{k+1})^m$, and U_m has precisely $\binom{l}{m}$ elements, so $\sigma(U_m) \leq \binom{l}{m}/(p_{k+1}{}^m)$.

Hence

$$\sigma(U) \leq \sum_{m=1}^{l} \binom{l}{m} \frac{1}{{p_{k+1}}^m} = \left(1 + \frac{1}{p_{k+1}}\right)^l - 1.$$

Since at least every second natural number is composite, we have

$$\begin{aligned} l &< \frac{1}{2}\left((r+1)^2 - r^2\right) \\ &< r. \end{aligned}$$

Also, $p_{k+1} > r^2$ by definition. So we have

$$\begin{aligned} \sigma(U) &\leq \left(1 + \frac{1}{p_{k+1}}\right)^r - 1 \\ &\leq \left(1 + \frac{1}{r^2}\right)^r - 1 \\ &= O\left(\frac{1}{r}\right). \end{aligned}$$

Let $V = S - U = \{n \in \mathbb{N} \,|\, n \text{ is square-free, and all prime factors of } n \text{ are } \leq r^2\}$. We claim $\sigma(V)$ is $O(\log r)$. To see this, choose a constant C greater than $\sum_{q=1}^{\infty} \frac{1}{q^2}$. Then, with p denoting an arbitrary prime number, we have[3]

$$\begin{aligned} C &\geq \prod_{p \leq r^2} \left(1 - \frac{1}{p^2}\right) \\ &= \prod_{p \leq r^2} \left(1 + \frac{1}{p}\right) \prod_{p \leq r^2} \left(1 - \frac{1}{p}\right) \\ &\sim \prod_{p \leq r^2} \left(1 + \frac{1}{p}\right) \frac{e^{-\gamma}}{\log r^2} \end{aligned}$$

(Here γ is Euler's constant; the fact that $\prod_{p \leq q} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log q}$ is proved in Hardy and Wright [5], page 351.) Hence

$$\prod_{p \leq r^2} \left(1 + \frac{1}{p}\right) = O(\log r).$$

But $\sigma(V) = \prod_{p \leq r^2} \left(1 + \frac{1}{p}\right)$, so the claim is proved.

Now from the definitions of S, U, and V, it is clear that $S = UV$. (That is, $S = \{n \,|\, n = uv, u \in U, v \in V\}$.) And because each element of U is coprime to each

[3] Dr Wayne Walker suggested this approach.

element of V, this means $\sigma(S) = \sigma(U)\,\sigma(V)$. Thus

$$\begin{aligned}\sigma(S) &= O\Big(\frac{1}{r}\Big)\,O(\log r)\\ &= O\Big(\frac{\log r}{r}\Big),\end{aligned}$$

proving the lemma. □

The next theorem now follows almost verbatim from Davenport:

Theorem 3. *(RDF) holds almost everywhere.*

Proof. First we claim it will be enough to show $T_{r^2}(\theta) \to 0$ as $r \to \infty$ for almost all θ. For if $r^2 \le n \le (r+1)^2$, we have

$$\begin{aligned}|T_n(\theta) - T_{r^2}(\theta)| &= \left| \sum_{k=t_{r^2}+1}^{t_n} \frac{\mu(s_k)}{s_k}\{s_k\theta\} \right|\\ &\le \sum_{k=t_{r^2}+1}^{t_{(r+1)^2}} \frac{1}{s_k}.\end{aligned}$$

This sum is precisely the sum of reciprocals of the set

$$\left\{\begin{matrix} n \in \mathbb{N} \,|\, n \text{ is square-free, } n \text{ has at least one prime} \\ \text{factor} > r^2, \text{ and } n \text{ has no prime factor} \ge (r+1)^2 \end{matrix}\right\},$$

which, by the previous lemma, is $O(\log r/r)$. So

$$|T_n(\theta) - T_{r^2}(\theta)| = O\Big(\frac{\log r}{r}\Big),$$

proving the claim.

Let E_r denote the set of points where

$$|T_{r^2}(\theta)| > \frac{1}{\log r}.$$

Fix $\epsilon \in (0, \frac{1}{2})$. By Lemma 1, the Lebesgue measure of E_r is bounded by

$$\lambda(E_r) < C\frac{\log r}{r^{2-2\epsilon}},$$

for some constant C. Let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{r=n}^{\infty} E_r.$$

Then

$$\lambda(E) \leq \lim_{n\to\infty} \sum_{r=n}^{\infty} C \frac{\log r}{r^{2-2\epsilon}}$$
$$= 0.$$

But any point θ for which $T_{r^2}(\theta)$ does not tend to zero belongs to infinitely many of the E_r, and is therefore in E. This completes the proof. □

Finally we prove the promised result, that the Summability by Primes formula holds for a large class of step functions.

Let $f_b(\theta)$ be the even step function given by

$$f_b(\theta) = \begin{cases} 1 - \frac{b}{\pi} & \text{if } |\theta| < b, \\ \frac{1}{2} - \frac{b}{\pi} & \text{if } \theta = \pm b, \\ -\frac{b}{\pi} & \text{if } b < |\theta| \leq \pi. \end{cases}$$

(So $f_b(\theta)$ is just the normalisation of the function equalling 1 when $|\theta| < b$, $1/2$ when $|\theta| = b$, and 0 elsewhere. The constant b/π has been subtracted so that $\int_{-\pi}^{\pi} f_b(\theta)\, d\theta$ is zero.) A quick calculation shows that the nth Fourier coefficient of f_b is given by

$$a_n = \frac{2}{n\pi} \sin(bn).$$

Theorem 4. *The Summability by Primes formula is valid for $f_b(\theta)$, for almost all values of $b \in [0, \pi]$.*

Proof. Let

$$E_n = \left\{ \begin{matrix} b \in [0, 2\pi] \text{ such that (SBP) does } \textit{not} \text{ calculate} \\ \text{the } n\text{th Fourier cosine coefficient of } f_b \text{ correctly} \end{matrix} \right\}$$

We claim $\lambda(E_n) = 0$. To see this, first suppose b is such that the rearranged Davenport formula holds at $\theta = \frac{nb}{2\pi}$. That is,

$$\lim_{q\to\infty} \sum_{k=1}^{\infty} \delta_k^q \frac{\mu(k)}{k} \left\{ \frac{kbn}{2\pi} \right\} = -\frac{1}{\pi} \sin(bn).$$

Under this assumption, we want to show that for the function $f_b(\theta)$,

$$(*) \qquad a_n = \lim_{q\to\infty} \sum_{k=1}^{\infty} \delta_k^q \mu(k) S(nk).$$

As calculated by Schiff and Walker [3], we have $S(nk) = -\frac{2}{nk}\left\{\frac{bnk}{2\pi}\right\}$. Thus

$$\begin{aligned}\text{RHS of } (*) &= -\frac{2}{n}\lim_{q\to\infty}\sum_{k=1}^{\infty}\delta_k^q\frac{\mu(k)}{k}\left\{\frac{bnk}{2\pi}\right\}\\ &= \frac{2}{n\pi}\sin(bn)\\ &= a_n\end{aligned}$$

So (SBP) calculates a_n correctly provided the rearranged Davenport formula holds at $\theta = \frac{nb}{2\pi}$. By the previous theorem, this occurs at almost all values of b. Hence $\lambda(E_n) = 0$, as claimed.

Now set

$$\begin{aligned}E &= \begin{Bmatrix} b \in [0, 2\pi] \text{ such that (SBP) does not calculate}\\ \text{some Fourier cosine coefficient of } f_b \text{ correctly}\end{Bmatrix}\\ &= \bigcup_{n=1}^{\infty} E_n\end{aligned}$$

Then we have $\lambda(E) \leq \sum_{n=1}^{\infty}\lambda(E_n) = 0$, proving the theorem. □

3. A special case, and a formula of Euler

When $b = \frac{\pi}{2}$ we can directly calculate the result of applying the Summability by Primes formula to f_b. Write

$$f(\theta) = f_{\frac{\pi}{2}}(\theta) = \begin{cases}\frac{1}{2} & \text{if } |\theta| < \frac{\pi}{2},\\ 0 & \text{if } |\theta| = \frac{\pi}{2},\\ -\frac{1}{2} & \text{if } \frac{\pi}{2} < |\theta| \leq \pi.\end{cases}$$

Then $a_n = \frac{2}{n\pi}\sin\frac{n\pi}{2}$; in particular, $a_1 = \frac{2}{\pi}$. To distinguish between these (true) Fourier coefficients and those calculated by (SBP), define

$$\hat{a}_n = \lim_{q\to\infty}\sum_{k=1}^{\infty}\delta_k^q\mu(k)S(nk).$$

As remarked in the proof of the last theorem, a calculation in Schiff and Walker [3] gives

$$\begin{aligned}S(k) &= -\frac{2}{k}\left\{\frac{k}{4}\right\}\\ &= \begin{cases}0 & \text{if } k \text{ even},\\ \frac{1}{2k} & \text{if } k \equiv 1 \mod 4,\\ -\frac{1}{2k} & \text{if } k \equiv -1 \mod 4.\end{cases}\end{aligned}$$

For convenience define $r(k)$ by

$$r(k) = \begin{cases} 0 & \text{if } k \text{ even}, \\ 1 & \text{if } k \equiv 1 \mod 4, \\ -1 & \text{if } k \equiv -1 \mod 4. \end{cases}$$

Then $S(k) = \frac{1}{2k} r(k)$. So we can calculate $\hat{a}_1$:

$$(*) \qquad \hat{a}_1 = \lim_{q\to\infty} \sum_{k=1}^{\infty} \delta_k^q \mu(k) \frac{r(k)}{2k}.$$

But if k is odd and square-free we have the following facts:

$$r(k) = \prod_{\substack{p \text{ a prime} \\ \text{factor of } k}} r(p),$$

$$k = \prod_{\substack{p \text{ a prime} \\ \text{factor of } k}} p,$$

$$\mu(k) = \prod_{\substack{p \text{ a prime} \\ \text{factor of } k}} (-1).$$

So we can rewrite $(*)$ as

$$\begin{aligned} \hat{a}_1 &= \lim_{q\to\infty} \frac{1}{2} \sum_{\substack{k \text{ odd and} \\ \text{square-free}}} \delta_k^q \prod_{\substack{p \text{ a prime} \\ \text{factor of } k}} \frac{-r(p)}{p} \\ &= \frac{1}{2} \lim_{q\to\infty} \prod_{n=2}^{q} \left(1 - \frac{r(p_n)}{p_n}\right) \\ &= \frac{1}{2} \prod_{n=2}^{\infty} \left(1 - \frac{r(p_n)}{p_n}\right). \end{aligned}$$

Therefore

$$\frac{1}{2\hat{a}_1} = \prod_{n=2}^{\infty} \left(1 - \frac{r(p_n)}{p_n}\right)^{-1}.$$

But we also have

$$\begin{aligned} \frac{1}{2a_1} &= \frac{\pi}{4} \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \\ &= \sum_{k=1}^{\infty} \frac{r(k)}{k}. \end{aligned}$$

So $a_1 = \hat{a}_1$, and hence (SBP) calculates a_1 correctly for $f(\theta)$, if and only if

$$\text{(E)} \qquad \sum_{k=1}^{\infty} \frac{r(k)}{k} = \prod_{n=2}^{\infty} \left(1 - \frac{r(p_n)}{p_n}\right)^{-1}.$$

The first person to consider this formula was probably Leonhard Euler; it appears on page 244 of his *Introduction to Analysis of the Infinite* [6] along with a plethora of other formulas for infinite products involving prime numbers. However, Euler's method of proof cannot be patched up to meet modern standards of rigour, since his approach involves rearranging a series that is not absolutely convergent[4]. Fortunately for us, Euler's intuition did not mislead him on this occasion: Edmund Landau gives a correct proof using a more sophisticated technique in his *Handbuch der Lehre von der Verteilung der Primzahlen* [7], pages 446–9[5]. So we conclude that $\hat{a}_1 = a_1$.

But almost identical calculations show the same is true for $\hat{a}_n$ when n is odd, and it is very simple to see $\hat{a}_n = a_n = 0$ for even n. So in fact, we have proved that the Summability by Primes formula correctly calculates all Fourier coefficients of $f_{\frac{\pi}{2}}(\theta)$.

Many thanks must go to Dr Wayne Walker, who provided me with this problem, his ideas, and a great deal of enthusiasm and encouragement.

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[4] *Introduction to Analysis of the Infinite* contains a striking mixture of intuition, ingenuity, and sheer calculation—something modern students of analysis rarely experience. Presumably this is what led to Andre Weil's dictum (quoted by Blanton in his introduction to [6]) to the effect that "our students of mathematics would profit much more from a study of Euler's *Introductio in Analysin Infinitorum*, rather than of the available modern textbooks."

[5] I am particularly grateful to Professor Paul Bateman of the Univeristy of Illinois for providing this reference.