Helicity conservation under Reidemeister moves

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We discuss a connection between two fields that appear to have little in common: plasma physics and mathematical knot theory. Plasma physicists are interested in studying helicity conservation in magnetic flux ropes and knot theorists commonly consider "Reidemeister moves," transformations that preserve a property called "knottedness." To study the tangling, twisting, and untwisting of magnetic flux ropes, it is helpful to know which topological transformations conserve helicity. Although the second and third types of Reidemeister moves applied to a magnetic flux rope clearly conserve the helicity of the flux rope, the first type of Reidemeister move appears to be in conflict with helicity conservation. We show that all three Reidemeister moves conserve helicity in magnetic flux ropes. © *2006 American Association of Physics Teachers.*
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I. INTRODUCTION

It is gratifying when a concept or theory can be applied to explain different phenomena such as the application of Gauss's law to find the $1/r^2$-dependence of the electric field of a point charge, the light intensity emitted by a star, the gravitational field of a planet, and the sound intensity emitted by an idling lawn mower. It is even more satisfying if we can establish links between fields of study that are apparently disjoined. In this paper we discuss a connection between plasma physics and mathematical knot theory.

Some plasma physicists are interested in magnetic flux ropes, which are bundles of magnetic field lines that occur, for example, in solar prominences in the solar corona,[1–3] in magnetic clouds in the interplanetary plasma,[4] and in the Venusian ionosphere.[5] Flux ropes in the Sun affect daily life on Earth: As a result of instabilities, coronal mass ejections spew plasma toward the Earth's magnetosphere, resulting in interruptions in satellite communications and power outages. An increased understanding of the motion and stability of magnetic flux ropes in the Sun might aid space weather forecasters to predict coronal mass ejections. Of particular interest is the reconnection of flux ropes and the helicity of these topological structures.

Knot theory investigates topological configurations consisting typically of one-dimensional strings, with application to ropes, garden hoses, and extension cords, as well as many areas of science. In physics, the tools of knot theory, including signed planar graphs, Reidemeister moves, and the Arf invariant,[6] greatly simplify the calculation of the partition function of the Ising model of ferromagnetism. The theory of the stability of knots has inspired a new design of quantum computers.[7] In chemistry, the behavior of chemical bond structures relates closely to knot theory, which assists in the synthesis of knotted molecules and understanding of topological stereoisomers.[6,8] In biological applications, knot theory has been useful in studying knotted and twisted DNA structures, allowing the exploration of the functions of different enzymes that assist in DNA transcription, recombination, and replication.[6,8–11]

In this paper we show that the study of magnetic field topologies can be facilitated by knot theory. Helicity, a measure of the knottedness, linkedness, and twistedness of magnetic field topologies, is conserved under physical deformations due to bending and twisting and even under such extreme circumstances as magnetic field line reconnection.[12] Topologically, conservation of helicity is related to conservation of knottedness in knot theory. The question arises if the helicity of magnetic structures is also conserved under the types of topological transformations called Reidemeister moves in knot theory, which do conserve knottedness. In this paper we solidify the link between plasma physics and knot theory by showing that helicity is conserved under all three types of Reidemeister moves.

II. HELICITY

Helicity is defined as the integral over all space

$$H \equiv \int \boldsymbol{A} \cdot \boldsymbol{B} dV, \tag{1}$$

where $\boldsymbol{A}$ is the magnetic vector potential, which is related to the magnetic field $\boldsymbol{B}$ by $\boldsymbol{B}=\nabla \times \boldsymbol{A}$. Although helicity is often associated with linked flux tubes, there are three different disguises that helicity can take: knottedness, linkedness, and twistedness.[8,12] Helicity can be measured in a simple manner using a "helicity meter,"[12] two pencils attached to each other at right angles as shown in Fig. 1.

The "helicity meter" applies to a projection (see Sec. III) of magnetic flux ropes by suitably aligning two pencils at a crossing of two flux ropes or the crossing of a flux rope with itself. We start at a flux tube crossing and align the top pencil so that it points in the same direction as the upper strand of the diagram. If the lower strand follows the same direction as the bottom pencil, then this crossing contributes an amount $+\Phi^2$ to the helicity. If the lower strand has the opposite direction, then it contributes an amount $-\Phi^2$ to the helicity.[13] Here Φ represents the magnetic flux in the flux rope, defined as

$$\Phi \equiv \int \boldsymbol{B} \cdot d\boldsymbol{S}, \tag{2}$$

where $d\boldsymbol{S}$ is the normal area vector to the cross-sectional surface area element, dS. The sum of all signed helicity contributions at each crossing results in the total magnetic helicity of the structure, assuming that there is no internal twisting. We encourage the reader to verify the total helicity contained in the sample configurations displayed in Fig. 2.

Fig. 1. A helicity meter consisting of two pencils.

III. REIDEMEISTER MOVES

Reidemeister moves are used to manipulate projections of mathematical knots. A mathematical knot is a closed curve in three dimensions.[14] A projection of a knot is a two-dimensional representation of a three-dimensional knot. Just as there are infinitely many ways of viewing a three-dimensional object, there are infinitely many projections of a single knot. The Reidemeister moves shown in Fig. 3 convert one projection of a particular knot into another projection of the same knot leaving the degree of knottedness invariant. These moves do not alter the knot, simply the way one views it.[14,15]

Reidemeister move I takes a segment of a knot projection with no local crossings and twists it so that there is a new crossing in the projection, but does not change the fundamental properties of the knot itself. Conversely, removing a crossing of the proper form is also described by move I. Reidemeister move II allows one segment of a projection to pass over (or under) another segment, and Reidemeister move III allows a nearby segment to pass over (or under) a crossing. With these three sets of maneuvers, a projection of a mathematical knot can be transformed into another projection of an equivalent knot.[15]

IV. THE APPARENT PROBLEM

The Reidemeister moves displayed in Fig. 3 affect small portions of a larger knot. We assume the part of the knot that is not shown remains unaltered. At this point, we again encourage the reader to use the helicity meter to determine the helicity of the three pairs of rope projections shown in Fig. 3. The projections must have a relative directionality associated with them. For this purpose assign a consistent direction to all strands involved before measuring the helicity. Reversing the direction associated with all strands preserves the helicity. This exercise will reveal a problem concerning the conservation of helicity.

Let us first look at Reidemeister moves II and III. The knot portions IIa and IIb contain the same helicity, $H=0$, because there are no crossings in IIa and the two crossings in IIb contribute $+\Phi^2$ and $-\Phi^2$ to the total helicity of the knot. Similarly, the net contribution of the three crossings in IIIa are $+\Phi^2$ (or $-\Phi^2$), depending on the direction of the magnetic field, just as the configuration in IIIb contributes the same value, $+\Phi^2$ (or $-\Phi^2$). Thus, Reidemeister moves II and III undisputedly conserve helicity.

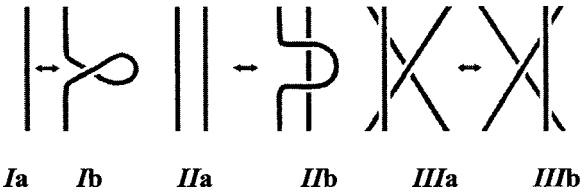

Fig. 3. The Reidemeister moves alter knot projections but not the knot itself.

The situation in Reidemeister move I does not seem as straightforward. Although the segment in Ia has no crossings and consequently contributes no helicity, the section in Ib contributes $+\Phi^2$ (or $-\Phi^2$) to the helicity. Is this a case where helicity is not conserved?

V. THE SOLUTION

It is important to note that mathematical knots are infinitesimally thin, as are magnetic field lines. Because we are interested in the helicity of magnetic flux ropes that have a finite thickness, we must approach knot theory in a slightly different manner. Visualizing knots as ribbons provides a way to add thickness to knots, making knot theory and Reidemeister moves more accessible to magnetic flux ropes.

We present here two independent solutions to this apparent problem, a purely topological demonstration employing ribbons and an analytic proof. Due to its elegance and simplicity, we will discuss the topological verification first.

During a Reidemeister-I move, a magnetic flux rope gains a crossing, adding an amount of $\pm\Phi^2$ to the total helicity. At the same time the flux rope also receives an internal twist. It turns out that this twist contains an amount of helicity equal to $\mp\Phi^2$.

To understand how a twist can contribute to the helicity, consider a ribbon containing a 360° twist. Figure 4 illustrates that a ribbon with a 360° twist can be converted to an untwisted ribbon with a single crossing. The reader may readily verify this property with a paper ribbon.

Now imagine performing a Reidemeister-I move on an untwisted, unknotted portion of a ribbon [see Fig. 5(a)], while keeping the remainder of the ribbon unchanged. We will end up with a ribbon that has a 360° twist and one crossing [see Fig. 5(b)]. However, as just demonstrated, the 360° twist can be turned into a second crossing [see Fig. 5(c)]. Now we have an untwisted ribbon with two crossings.

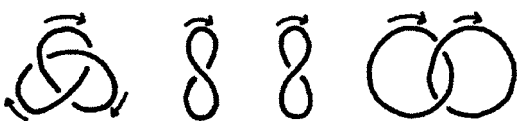

Fig. 2. Calculation of the helicity contained in each configuration using the helicity meter, assuming each structure has no internal twisting. The reader can verify that the four knot projections contain $H=-3\Phi^2$, $H=+\Phi^2$, $H=-\Phi^2$, and $H=-2\Phi^2$, respectively, from left to right.

Fig. 4. A ribbon with a 360° twist is equivalent to an untwisted ribbon with one crossing. Shown here is a counterclockwise twist containing helicity $-\Phi^2$ when measured with the helicity meter of Fig. 1. A clockwise twist contains helicity $+\Phi^2$.

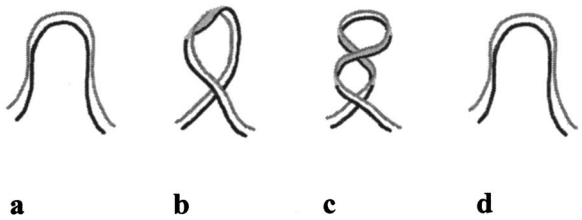

Fig. 5. Topological sequence showing how a Reidemeister-I move followed by a twist-to-crossing conversion results in a Reidemeister-II move, demonstrating conservation of helicity. The black and gray curves designate the two edges of the ribbon.

Because we have already verified that Reidemeister move II conserves helicity, this provides the proof that Reidemeister move I conserves helicity.

For the analytic proof we will divide the original flux rope first into two flux ropes of flux $\Phi/2$, then into three flux ropes of flux $\Phi/3$, and finally into n flux ropes of flux Φ/n. In each case we will show the conservation of the total helicity under a Reidemeister-I move. The division of the rope into two and three parts is simply for illustrative purposes while the n divisions show the validity in general. As seen in the topological proof, a Reidemeister-I move adds a crossing (contributing Φ^2) and a 360° twist (contributing $-\Phi^2$), resulting in no net helicity change.[16]

By performing the Reidemeister-I move, we produce the anticipated crossing, which contributes $(\Phi/2+\Phi/2)^2=\Phi^2$ to the total helicity.[17] Furthermore, we now observe that the $\Phi/2$-flux ropes twist once around each other. If we project this twisted portion onto a plane, the twist [cf. the top left section of the ribbon shown in Fig. 5(b)] is equivalent to two crossings of the $\Phi/2$-flux ropes contributing $-(\Phi/2)^2$ and $-(\Phi/2)^2$, that is, $-2(\Phi/2)^2$ to the helicity. It appears the total helicity now no longer adds up to zero. However, a closer inspection of the $\Phi/2$-flux ropes reveals that both $\Phi/2$-flux ropes contain internal twists of $-(\Phi/2)^2$ each, again bringing the total helicity to zero.

The reader may want to split the original flux rope into three equal pieces and perform the Reidemeister-I move. By adding the resulting crossings and twists, the reader will again find Φ^2 for the main crossing, then each of the three $\Phi/3$-flux ropes crossing the other two [amounting to $-6(\Phi/3)^2$], and each $\Phi/3$-flux rope containing $-(\Phi/3)^2$ in internal twist helicity, contributing $-3(\Phi/3)^2$ to the total helicity. The interaction between the internal twisting and the $\Phi/3$-flux ropes crossing each other is displayed in Fig. 6.

The generalization to n individual strands of flux Φ/n similarly yields a main crossing helicity contribution of Φ^2, n individual strands crossing their $n-1$ partners [contributing $-n(n-1)(\Phi/n)^2$], and each Φ/n-strand containing internal twist helicity of $-(\Phi/n)^2$ [contributing $-n(\Phi/n)^2$]. As in the $n=2$ and $n=3$ cases, the total helicity following the Reidemeister-I move is again zero,

Let us investigate the helicity contributions as n approaches infinity. The helicity contribution from the crossing remains Φ^2. The helicity contribution from each strand crossing the other strands approaches $-\Phi^2$, and the helicity contribution from internal twisting goes to zero,

$$\lim_{n\to\infty}[\Phi^2]=\Phi^2, \tag{3}$$

$$\lim_{n\to\infty}\left[-n(n-1)\left(\frac{\Phi}{n}\right)^2\right]=-\Phi^2, \tag{4}$$

$$\lim_{n\to\infty}\left[-n\left(\frac{\Phi}{n}\right)^2\right]=0. \tag{5}$$

It makes sense that the internal twist helicity contribution approaches zero because the infinitesimal flux rope represents a single magnetic field line, which cannot have internal twist because magnetic field lines have no thickness. By investigating magnetic flux ropes as structures with finite thickness and reevaluating the calculations as the sub-flux ropes become infinitesimally thin, we see that this analysis is consistent with both the behavior of the flux rope structure as well as the properties of individual magnetic field lines. Thus, helicity is conserved under all three Reidemeister moves.

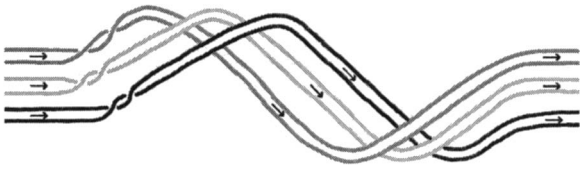

Fig. 6. $\Phi/3$-flux ropes after performing a Reidemeister-I move. The reader can use the pencil helicity meter to verify the helicity contributions due to the crossings.

VI. CONCLUSION

Because helicity measures the knottedness, linkedness, and twistedness of magnetic field lines, it is natural to associate knot theory with helicity conservation. The topological and analytic proofs outlined here show that the helicity of a magnetic flux rope is conserved if it is subject to any one of the three Reidemeister moves. In particular, we have shown that the apparent violation of helicity conservation in Reidemeister move I is only a violation at first glance. It turns out that the increase (or decrease) in helicity gained by the additional crossing resulting from a Reidemeister move I is offset by the helicity decrease (or increase) from the internal twisting of each magnetic flux rope as well as the twisting of each field line around every other field line. Therefore, the application of Reidemeister moves for the sake of determining the helicity of a magnetic flux rope may become a useful tool in plasma physics.

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[16]The crossing introduces a $+\Phi^2$ flux contribution for a crossing with the directionality: crossed arrows. Thus, a Reidemeister move that creates this type of main crossing will produce an internal 360° twist that contributes $-\Phi^2$ to the helicity. If the Reidemeister move creates a crossing like the crossed arrows, the crossing will contribute $-\Phi^2$ to the helicity and the internal twist contributes $+\Phi^2$. Without loss of generality, we assume the directionality of the first diagram for the analytic proof.
[17]Figures 5(a) and 5(b) demonstrate the interaction between the two $\Phi/2$-flux ropes (the black curve and the gray curve). These two flux ropes together represent the original Φ-flux rope.