IOP PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **29** (2008) 143–153 doi:10.1088/0143-0807/29/1/015

Synchronization of oscillators: an ideal introduction to phase transitions

L Q English

Department of Physics and Astronomy, Dickinson College, Carlisle, PA 17013, USA

Received 12 September 2007, in final form 23 October 2007
Published 17 December 2007
Online at stacks.iop.org/EJP/29/143

Abstract

The spontaneous synchronization of phase-coupled, non-identical oscillators is explored numerically via the famous Kuramoto model. The conditions for synchronization are examined as a function of the coupling network. I argue that such a numerical exploration provides a feasible way to introduce the topic of phase transitions early in the physics curriculum. Furthermore, this approach can be used to familiarize undergraduate students with the notions of emergence and universality.

1. Introduction

The physics of phase transitions is not usually taught in the undergraduate physics major. Sometimes, time-permitting, it is introduced at the end of a statistical mechanics course, but most students do not broach this vast and important subject until their graduate studies. The underlying assumption is that the topic is too technical and sophisticated to be introduced earlier in the physics major. It is true that the journey towards understanding magnetic phase transitions is somewhat arduous, leading through the Heisenberg model, Boltzmann statistics and mean-field approximations. However, the rewards of an early introduction to phase transitions are two-fold. First, phase transitions are ubiquitous in nature, ranging from the gas–liquid–solid transformations, to magnetic and order–disorder transitions in solids. Secondly, they are among the most fascinating phenomena encountered in physics, involving global reorganizations, and they exemplify two important philosophical notions in contemporary science—emergence and universality.

In this paper, I describe how the well-known Kuramoto model, which exhibits what one might refer to as a temporal phase transition, can be used to introduce the subject of phase transitions in a way that is accessible even to second-year students. Not only is the necessary mathematical machinery less demanding than that of standard treatments, students also encounter problems that are presently of much scholarly interest and that are interdisciplinary in nature. Thus, as an added benefit, students learn about synchronization of biological oscillators and gain an appreciation for the wide range of applicability of statistical mechanics.

0143-0807/08/010143+11$30.00

The mathematical biologist Art Winfree [2, 3] in the mid-1960s developed the first model to explain the growing body of evidence for mass-synchronization in biological systems. Some examples would include the synchronous flashing of fireflies, chirping of crickets, the pacemaker cells in the heart, or the neural cluster responsible for the circadian rhythm [3–7]. Later, Y Kuramoto simplified and streamlined Winfree's original model of interacting oscillators and formulated a system of nonlinear differential equations that he could solve analytically [1]. Since then, his model has been subjected to many theoretical studies [8, 9] and has been widely applied to problems in physics, chemistry, biology, ecology and sociology. Examples from physics include physical pendula [10], laser arrays [11], Josephson junctions [13] and fluid systems [12]. The model itself is very generic, but it captures the intriguing phenomenon of spontaneous synchronization.

This paper aims to show that a numerical exploration of the Kuramoto model can be used to familiarize students with phase transitions, as well as the important philosophical notions of emergence and universality. I argue that the latter goal is practically achieved by exploring the role of the underlying coupling network between oscillators. Recently, the role of the network in the collective behaviour of coupled oscillators, each with its individual characteristics, has received much attention [14–17]. I will concentrate on the simple globally-connected (or all-to-all) network and random networks, and I will briefly discuss recent results on small-world and scale-free random networks, as well as lattices. The critical point of the phase transition, and even its existence, depends greatly on the global properties of the network. On the other hand, the critical point is remarkably insensitive to the local details of the coupling.

2. The Kuramoto model

The Kuramoto model presumes a collection of limit-cycle oscillators best visualized as little dots moving endlessly around on a unit circle. The dots going around the circle represent the oscillators moving through the phases of their cycles. Every oscillator is presumed to have an intrinsic, in-built frequency, so that the dots move around the circle at varying speeds. In addition, the oscillators influence one another, and they do this by pulling on their respective frequencies. Thus, the instantaneous frequency of a given oscillator is determined both by its intrinsic frequency as well as by the net pull of all the other oscillators. That pull depends only on the relative phases of the other oscillators, or the positions of all the other dots on the unit circle. This coupling can be thought of as a mean field that acts to pull the oscillators into the common centre. Mathematically, the evolution of all N-oscillators within the Kuramoto model are given by

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i). \tag{1}$$

Here, θ_i is the phase of the ith oscillator, ω_i its intrinsic frequency, and K is the coupling constant determining by how much the intrinsic frequency can be modulated. The dynamical system of equation (1) thus governs the evolution of our oscillators. It is easy to see that the sine-function coupling accomplishes the desired effect of encouraging clustering: if the jth oscillator is ahead in phase of the ith oscillator, then the former oscillator is slowed down whereas the latter oscillator is sped up by the interaction. Thus, there is the tendency to converge in phase. A convenient measure of the degree of synchronicity is r defined in the expression,

$$r\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\theta_j(t)}. \tag{2}$$

The square of this r is thus evaluated as

$$r^2 = \frac{1}{N^2}\left[\left(\sum_j \cos(\theta_j)\right)^2 + \left(\sum_j \sin(\theta_j)\right)^2\right] \tag{3}$$

and this expression will serve as our degree of synchronicity. Clearly, r^2 ranges from zero to one. When the oscillators are randomly spread out in phase, dots cover the entire unit circle, and the sum of both cosines and sines tends to zero. In the other extreme, when all oscillators are in-phase, $\sum_i \cos(\theta_i) = N\cos(\theta)$ and $\sum_i \sin(\theta_i) = N\sin(\theta)$, so that $r^2 = 1$.

The intrinsic frequencies, ω_i, are generated by a random number generator with a Gaussian probability distribution. Applied to biological oscillators, this frequency distribution of individual cells reflects the inevitable variability found in nature; no cell is exactly identical to another. Equation (1) reflects the all-to-all coupling of the Kuramoto model, and it can be demonstrated that this property makes it a mean-field model. In a mean-field model, any particular oscillator is only sensitive to average, global quantities of the collection of oscillators; the detailed configuration does not matter. One can show that equation (1) reduces to

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + Kr\sin(\psi - \theta_i), \tag{4}$$

where r and ψ are defined in equation (2).

This transformation of the governing equations speeds up the numerical simulations, and it allows for a theoretical prediction of the order parameter. Using a self-consistency argument very analogous to the one encountered in the ferromagnetic problem [18], one can obtain for the critical coupling,

$$K_c = \sqrt{\frac{8}{\pi}}\sigma \approx 0.16 \tag{5}$$

where σ is the standard deviation of the frequency distribution (here, $\sigma = 0.1$). Synchronization occurs for $K > K_c$. The mean-field approximation is usually made as a first line of attack in the study of phase transitions, but in the context of the Kuramoto model it holds true exactly.

Subsequently, we will depart from all-to-all coupling by considering oscillators connected via networks [19]. In other words, not all oscillators will be able to communicate with all others, but only those between which a link exists. Random networks are easily generated by randomly removing a certain percentage of links from the globally-connected (all-to-all) network. The network is introduced into our model simply by modifying equation (1):

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} G_{ij}\sin(\theta_j - \theta_i). \tag{6}$$

Here G_{ij} represents an N-by-N matrix with entries zero or one. We consider diagonally-symmetric matrices, which ensure that the connections between oscillators are bi-directional. The entries, G_{ij}, could also measure the strengths of connections, but here we restrict ourselves to zero and one. If G_{ij} represents a highly symmetric matrix, equation (6) explores the dynamics on regular lattices.

3. Numerical simulations

In part for pedagogical reasons, we decided to use an integrated software package called IgorPro[1], which is similar to IDL[2], as a programming platform as well as a data-visualization tool. The advantage is mainly the convenience of staying entirely within one environment for all aspects of this project. Furthermore, the programming is very straightforward and students can focus quickly on the physics (rather than on the idiosyncrasies of a particular language).

The main code is reproduced in the appendix. After defining variables and arrays (which are called 'waves' in Igor Pro), the oscillator frequencies are randomized using the in-built *gnoise* function which generates a random number according to a Gaussian probability density cantered at zero with a user-defined standard deviation. In all simulations, the mean frequency was set at 1 (although 0 would also work) and the standard deviation at 0.1. Thereafter, the oscillator phases are initialized. In the example shown, we chose to spread out the phases over the full unit circle (x is the array index here).

At the heart of the code is a numerical integration of equation (4) using the simple Euler method. Within the outer time-step loop, the first for-loop runs through the array index and computes the parameters r and ψ for a given oscillator configuration (see equation (3)). In the second loop, these two parameters are then used to update the phases of all oscillators.

3.1. The globally-connected network

We start with all oscillators coupled to all others. In figure 1, the 250 oscillators are represented as black dots going around in a circle. The angle in this polar plot simply indicates the phase within the cycle of a given oscillator, and the radius is the intrinsic frequency. The starting configuration is shown in figure 1(a). The oscillators are all at random phases with respect to one another, and they occupy a ring in this plot, indicating that there is a spread of intrinsic frequencies (about 1) present in this population. The subsequent panels show the resultant configuration after a long time has passed (1000 computational steps) for increasing values of the coupling constant K in equation (1). All dots move counter-clockwise.

In figure 1(b), the coupling constant, K, is set to 0.2, which is just slightly to the right of the critical point. The system then evolves into two components: the synchronized cluster and the unsynchronized background. The former is identified in the figure as the curved line (spiral segment), and the latter as the dots that do not fall on this line.

The coupling constant, K, is raised to 0.4 in figure 1(c), and we see that the unsynchronized component has vanished. All dots now fall on the spiral segment. This curve reveals the following principle of organization: those oscillators in the synchronized cluster that have an intrinsic frequency above the mean will lead in phase, and those with lower than average intrinsic frequencies will trail. This makes good intuitive sense, as the only thing holding the fast oscillators back is the combined pull of the slower ones, and vice versa. When the coupling constant is again raised to $K = 0.8$, we observe the cluster becoming tighter in phase in figure 1(d). Incidentally, the time evolution of this radial plot can be best visualized by making movies. This can be done in real computational time when using a software package, such as Igor Pro (see footnote 1), which supports programming as well as data manipulation and graphing. To view a few examples of movie-clips, see [20].

Figure 2 shows the time evolution of the degree of synchronization in the oscillator population, as defined in equation (3). This global property of the system is shown as a function of time and for a few different initial conditions. In figure 2(a), the coupling strength

[1] IGOR Pro, 2004 Version 5, WaveMetrics Inc.
[2] IDL, Research Systems Inc.

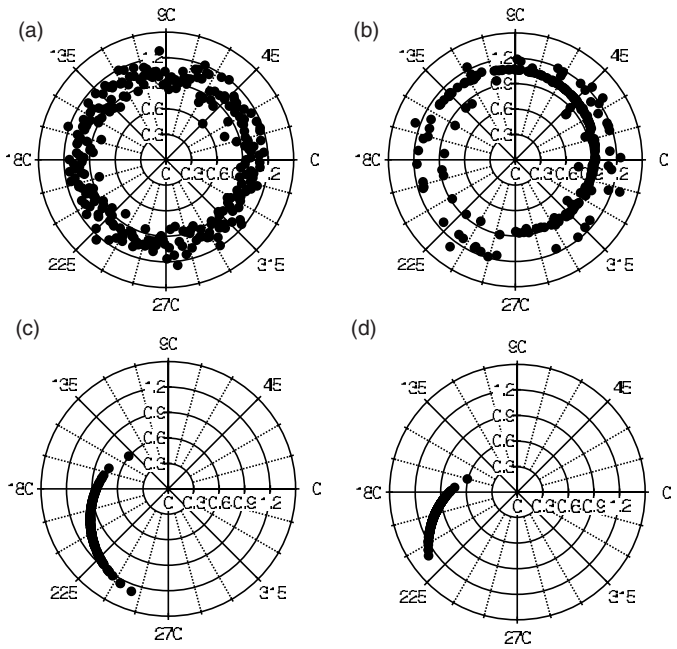

Figure 1. A polar plot of the distribution of oscillators. The azimuthal axis depicts the phase within the cycle and the radial axis the intrinsic frequency of the oscillator. Part (a) shows the initial configuration, characterized by phase incoherence and a Gaussian distribution of intrinsic frequencies (cantered around 1). The subsequent panels show the final state (after 1000 time steps) for increasing coupling constants.

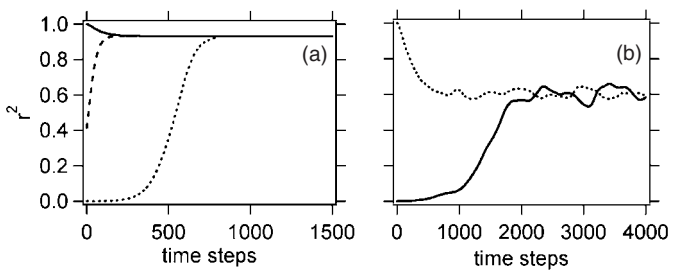

Figure 2. The degree of sync (defined by equation (3)) as a function of time for different initial conditions. (a) For a relatively high coupling value of 0.4, and three different initial conditions, ranging from all oscillators in phase (solid) to all of them out of phase (dotted). Note that the eventual degree of sync is the same. (b) A lower coupling constant ($K = 0.2$). Again, the average degree of sync reached eventually is not affected by the starting phases of the oscillators, but now there are clear fluctuations about the mean.

is set to $K = 0.4$, for which a high degree of synchronization is clearly achieved by the system. For the solid line, the system was initiated in complete sync by setting the starting phases of all oscillators to the same value (zero). This cluster then broadens in phase causing the degree of synchronization to drop slightly. The dashed line shows an initial condition where the phases of all the oscillators start out distributed over one half of the unit circle, whereas for the dotted line they are distributed over the entire unit circle corresponding to a completely

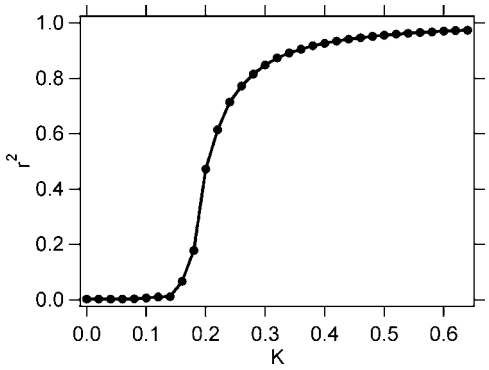

Figure 3. The bifurcation diagram of the Kuramoto model with 500 oscillators. The x-axis plots the coupling constant, and the y-axis the eventual degree of sync reached by the system. This was estimated by averaging the last 1000 time steps in 5000 time-step runs. We see that up until about $K = 0.14$ no appreciable synchronization can happen, at which point the degree of sync begins to rise from zero up towards one. The predicted critical point is $K = 0.16$.

unsynchronized state. The remarkable result is that the eventual degree of sync is strictly independent of how the oscillators start out.

In figure 2(b), the coupling parameter is reduced to $K = 0.2$. Here the system can still achieve some level of synchronization. However, it clearly takes the system longer to establish the partially synchronized state, and although the average level is still independent of the initial conditions, we observe strong fluctuations in these levels. Here the coupling is weak enough to permit the formation of an unsynchronized component. The fluctuations caused by this component grows as the coupling constant is reduced further, until all synchronization vanishes.

One might intuitively think that the degree of sync would depend on the coupling strength in a smooth, roughly linear manner, assuming that the smaller the coupling, the lower the eventual level of sync. This, however, is not the case. If we keep track of the eventual degree of sync as we increment the coupling strength, a graph like that shown in figure 3 is obtained. For low values of the coupling strength, the system is completely unable to synchronize even if we start out in perfect sync. No level of sync can be established over a range of coupling parameters extending up to the 'critical' coupling. At that point something new happens; the system is suddenly able to form a broad cluster that 'travels' together on the unit circle. Synchronization is now possible, and the degree of sync (i.e., the size of this cluster) depends sensitively on the coupling strength. The critical point in figure 3 is found to be at approximately $K = 0.15$, which is close (within the uncertainty) to the theoretical prediction in equation (5) of 0.16. For 500 oscillators, a graph like figure 3 can be generated on a standard PC within about 2 min.

Some readers will recognize this curve as the quintessential example of a second-order phase transition. The spontaneous magnetization of a ferromagnet as a function of temperature has a very similar shape. Above the Curie temperature, no zero-field magnetization can be achieved, and as we cool the ferromagnet through this critical point, the magnetization will rise and asymptotically approach the saturation magnetization. Since we are considering a mean-field theory (when dealing with all-to-all coupling), one might suspect that the order parameter, r, departs from zero near the critical point like a square-root, such that the critical exponent is one-half. Indeed, this exponent can be derived theoretically [18], but it is somewhat

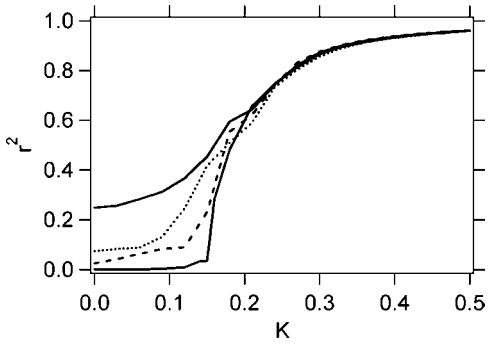

Figure 4. Dependence on system size. We see that the phase transition gradually emerges as the system size is increased from $N = 5$ (solid), to 15 (dotted), 50 (dashed) and 250 (solid).

difficult to confirm numerically, as the region in the close vicinity to the critical point requires large sample sizes to converge.

It should be noted that a very similar graph can be obtained for constant coupling strength but varying spread of oscillator frequencies. This is accomplished in practice simply by adjusting the standard deviation in the Gaussian distribution used to generate the population frequencies. Here, as the population spread or inhomogeneity is lowered through a critical point, the degree of sync branches off from zero. Not surprisingly, the two parameters, the coupling strength and the population spread, depend on one another at the critical point, as given by equation (5).

3.2. Towards emergence and universality

The emergent character of phase transitions is often under-appreciated [21]. Phase transitions, such as the ferromagnetic or superconducting transitions, are only observed in macroscopic samples. They lose their sharpness in nano-scale samples, such as quantum dots. The magnetic transitions disappear altogether in molecules containing just a few coupled spins. It is clear that the sharpness of the phase separation, as well as the robustness of these phases, is a macroscopic phenomenon and emerges in the limit as the sample size gets large.

In these simulations, emergence is fairly easy to demonstrate by simply examining populations of different numbers of oscillators. The results are shown in figure 4 which displays bifurcation graphs similar to figure 3, but now for different population sizes. The upper solid line corresponds to 5 oscillators, the dotted line to 15 oscillators, the dashed line is for 50 oscillators, and the lower solid line for 250 oscillators. It is evident that for only five oscillators a phase transition is not observed and no critical point can be identified. Instead, the degree of sync increases very smoothly with the coupling strength. To obtain this curve, many different runs with different populations were averaged together. In fact, the initial population of 250 oscillators was simply chopped up into groups of five, these subsets were then run separately, and the numerical results averaged. Interestingly, chopping and averaging does not at all yield the original result. As the number of oscillators is increased, a sharp transition gradually emerges.

Having explored the dependence on population size, let us now numerically investigate how the synchronization depends on coupling or network structure. For this purpose, we depart from the unrealistic scenario of all-to-all coupling, and start removing some links at random. In other words, we generate random networks of oscillator coupling and analyse the

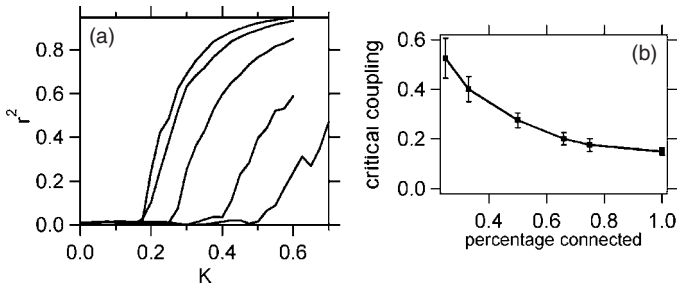

Figure 5. (a) The bifurcation diagram for Kuramoto oscillators on random networks. These networks are generated by randomly removing a certain fraction of connections between oscillators. Shown here are the results for different fractions of removed connections: moving from left to right, 0.25, 0.33, 0.5, 0.66, 0.75. (b) Estimates for the critical point, K_c, of the lattices from (a).

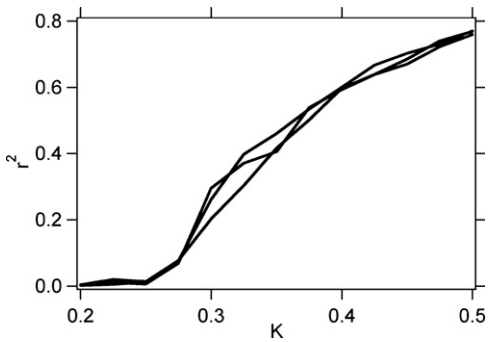

Figure 6. The phase diagram for three different random networks having only the degree of connectivity as their common feature.

effect on the phase transition. This can be done by assigning 0 or 1 with a predetermined probability to all the entries of G_{ij} in equation (6).

Figure 5(a) summarizes the data for various random networks of 250 oscillators. The left-most trace has a quarter of the links randomly removed, then progressing to the right, the traces correspond to networks with one-third, one-half, two-thirds and three-quarters of the links removed at random. The critical point is seen to shift to the right as the networks become more dilute. For all networks considered, we observe a fairly sharp onset of synchronization, but the curves seem to become increasingly shallow as the number of links in the network decreases.

Figure 5(b) shows an estimate for the critical coupling strength for each network. These estimates are based on a somewhat qualitative extrapolation of the order parameter to zero, and not on a more rigorous finite-size-scaling analysis [22, 23]. The decreasing slope of the critical point versus connectivity of the network is clearly discernable. This must be true, since the critical coupling must diverge to infinity as the network becomes less and less connected. From a larger perspective, figure 6 illustrates another aspect of emergence. Evidently, the phase transition depends not only on oscillator properties and the coupling mechanism, but also on the network that governs oscillator communication. The phase transition must be seen as a truly collective phenomenon.

In addition to emergence, we can begin to appreciate a second concept pertinent to phase transitions, namely universality. In figure 6, the three lines correspond to networks for which

half the possible links were removed. However, they are all very different networks. Their only common feature is a statistical property—the probability of a link being formed between pairs of oscillators. A local, node-by-node comparison between two such networks would uncover no similarity. Nevertheless, these traces are very close to one another indicating that it is precisely this global statistical property of the network that counts, rather than the microscopic structure of the network. For a more rigorous discussion of universality in the context of synchronization, see [24].

The counterpoints to random networks are regular lattices, the epitome of order. One of the simplest lattices we can study is the one-dimensional chain, with only nearest-neighbour couplings. Not surprisingly, since very few oscillators are now communicating directly with one another, global synchronization is very difficult to establish. In our simulations, we were unable to see a synchronized state even for extremely large coupling constants, confirming the general rule that phase transitions are impossible in one dimension. More precise studies [25] indicate that synchronization can only be confirmed for three- or higher-dimensional lattices. Neither random networks nor lattices, it turns out, are very good representations of real networks. To be sure, real networks do incorporate elements of randomness and structure, being both very connected as well as very clustered [15, 16]. Oftentimes, they are what is referred to as scale-free. In such networks, the number of nodes with N connections does not fall off exponentially fast (but rather algebraically slow) with N. Many actual networks observed in nature exhibit this characteristic [15, 16] due to basic principles at work in their formation and evolution. Recent studies show that the Kuramoto phase transition can vanish altogether in such networks (under certain conditions) [26], reminding us once more of the delicate influence of the network on the collective behaviour of synchronization.

4. Conclusions

The numerical data presented here explore some aspects of the synchronization of coupled oscillators, a topic of much interdisciplinary interest. This paper has focused on the 'temporal' phase transition intrinsic to the Kuramoto system and its dependence on the underlying coupling network. This dependence serves to highlight two pervasive notions in the study of phase transitions: emergence and universality.

I believe that these simulations can be used as a teaching tool in an introduction to phase transitions. They are fairly elementary and should be easy to reproduce by undergraduate students. When the code is incorporated into mathematical and data-manipulation software (such as Igor Pro), the oscillator dynamics can be visually examined in various parameter regimes. A numerical exploration, as outlined here, thus gives students a useful pedagogical tool in investigating first-hand the many different facets of phase transitions.

Appendix. The basic code in Igor Pro

```
function kuramoto(k)
variable k     //the coupling parameter

variable N, dt, tmax, s2,c2, psi, r, i, j, t
wave phi, omega, op     //waves are arrays

dt = 0.1 // set computation time step
```

```
N = 500 //set number of oscillators
tmax = 5000 //set iteration number

omega = gnoise(0.1)+1     //set oscillator frequencies
phi=6.28/N*x //set oscillator starting phases

for (t=0; t<tmax; t=t+1)
s2=0; c2=0;
for (i=0;i<N;i=i+1)     //computes the degree of sync
s2 = s2 + sin(phi[i])
c2 = c2 + cos(phi[i])
endfor
r= sqrt(c2^2+s2^2)/N
op[t] = r     // stores the degree of sync
psi= atan(s2/c2)     // computes the mean phase
if (c2<0)
psi= Pi+psi
endif
for (i=0;i<N; i=i+1)     // evolves all oscillators forward
phi[i] = phi[i] + ( omega[i] + (k*r*sin(psi-phi[i])) )* dt
endfor
DoUpdate     // updates all figures in real computational time
endfor
end
```

References

[1] Kuramoto Y 1984 *Chemical Oscillations, Waves and Turbulence* (Berlin: Springer)
[2] Winfree A T 1967 Biological rhythms and the behavior of populations of coupled oscillators *J. Theor. Biol.* **16** 15–42
[3] Winfree A T 1980 *The Geometry of Biological Time* (New York: Springer)
[4] Buck J and Buck E 1976 Synchronous fireflies *Sci. Am.* **234** 74–7
[5] Walker T J 1969 Acoustic synchrony—2 mechanisms in snowy tree cricket *Science* **166** 891
[6] Strogatz S H 2003 *Sync* (New York: Hyperion)
[7] Strogatz S H and Stewart I 1993 Coupled oscillators and biological synchronization *Sci. Am.* **269** 102–9
[8] Crawford J D 1994 Amplitude expansions for instabilities in populations of globally-coupled oscillators *J. Stat. Phys.* **74** 1047–84
[9] Acebron J A, Bonilla L L, Perez Vicente C J, Ritort F and Spigler R 2005 The Kuramoto model: a simple paradigm for synchronization phenomena *Rev. Mod. Phys.* **77** 137–85
[10] Pantaleone J 2002 Synchronization of metronomes *Am. J. Phys.* **70** 992–1000
[11] Kourtchatov S Y, Likhanskii V V, Naparotovich A P, Arecchi F T and Lapuchi A 1995 Theory of phase locking of globally coupled laser arrays *Phys. Rev.* A **52** 4089–94
[12] Paoletti M S, Nugent C R and Solomon T H 2006 Synchronization of oscillating reactions in an extended fluid system *Phys. Rev. Lett.* **96** 124101–1
[13] Wiesenfeld K, Colet P and Strogatz S H 1998 Frequency locking in Josephson junction arrays: connection with the Kuramoto model *Phys. Rev.* E **57** 1563–9
[14] Strogatz S H 2001 Exploring complex networks *Nature* **410** 268–76
[15] Watts J D 2003 *Six Degrees* (New York: Norton)

[16] Albert R and Barabasi A L 2002 Statistical mechanics of complex networks *Rev. Mod. Phys.* **74** 47–98
[17] Restrepo J G, Ott E and Hunt B R 2006 Emergence of coherence in complex networks of heterogenous dynamical systems *Phys. Rev. Lett.* **96** 254103
[18] Strogatz S H 2000 From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators *Physica* D **143** 1–20
[19] Ichinomiya T 2005 Bifurcation study of the Kuramoto transition in random oscillator networks *Prog. Theor. Phys.* **113** 1–14
[20] http://www.larsenglish.com/spontaneoussynchronization
[21] Laughlin R B 2005 *A Different Universe* (Cambridge: Basic Books)
[22] Marro J and Dickman R 1999 *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge: Cambridge University Press)
[23] Moreno Y and Pacheco A F 2004 Synchronization of Kuramoto oscillators in scale-free networks *Europhys. Lett.* **68** 603–9
[24] Wood K, Van den Broeck C, Kawai R and Lindenberg K 2006 Universality of synchrony: critical behavior in a discrete model of stochastic phase-coupled oscillators *Phys. Rev. Lett.* **96** 145701
[25] Sakaguchi H, Shinomoto S and Kuramoto Y 1987 Local and global self-entrainments in oscillator lattices *Prog. Theor. Phys.* **77** 1005–10
[26] Ichinomiya T 2004 Frequency synchronization in a random oscillator network *Phys. Rev.* E **70** 026116